

# Universal Luttinger Liquid Relations in the 1D Hubbard Model

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## Abstract

We study the 1D extended Hubbard model with a weak repulsive short-range interaction in the non-half-filled band case, using non-perturbative Renormalization Group methods and Ward Identities. At the critical temperature,  $T = 0$ , the response functions have anomalous power-law decay with multiplicative logarithmic corrections. The critical exponents, the susceptibility and the Drude weight verify the universal Luttinger liquid relations. Borel summability and (a weak form of) Spin-Charge separation is established.

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# 1 Main Results

## 1.1 Introduction

The Hubbard model, see *e.g.* [1], describing interacting spinning fermions on a lattice, plays the same role in quantum many body theory as the Ising model in classical statistical mechanics, that is it is the simplest model displaying many real world features: it is however much more difficult to analyze. While our understanding of the Hubbard model in higher dimensions at zero temperature is really poor (except for special choices of the lattice as in [2] and [3]), the situation is better in  $d = 1$ , when the model furnishes an accurate description of real systems, like quantum wires or carbon nanotubes [4].

The one dimensional Hubbard model (from now on the Hubbard model tout court) can be exactly solved by *Bethe ansatz*, as shown by Lieb and Wu [5]: the system is insulating in the half filled band case while it is a metal otherwise and the elementary excitations are *not* electronlike, a phenomenon which is nowadays called *electron fractionalization* [6]. Recently in [7] a strategy for a proof that the lowest energy state of Bethe ansatz form is really the ground state has been outlined (see also [8]). This method is however of little utility for understanding the asymptotic behavior of correlations; and does not apply in studying the ground state of a slight generalization of the model, the extended Hubbard model, that consists in replacing the local quartic interaction with a short-ranged one. Other approaches has been therefore developed to get more insights into the physical properties of the Hubbard model.

Under certain *drastic approximations*, like replacing the sinusoidal dispersion relation with a linear relativistic one and neglecting certain terms called *backward* and *umklapp* interactions (see after (2.25) for their definition), one obtains the *spinning Luttinger Model*, which is exactly solvable in a stronger sense, [9], [10]: all its Schwinger functions, at distinct points, can be explicitly computed. This model, describing interacting fermions, can be exactly mapped in a model of two kinds of *free bosons*, describing the propagation of charge or spin degrees of freedom and with *different* velocities (*spin-charge separation*); again, a phenomenon of electron fractionalization which has received a considerable attention in the context of high  $T_c$  superconductors [11]. Moreover, as in spinless Luttinger model, the correlations have a power law decay rate *with interaction dependent exponents*.

However, neglecting the lattice effects and backscattering or umklapp interactions is not safe, and indeed the mapping to free bosons is not expected to be true in the Hubbard model. A somewhat more realistic effective description can be obtained by including the backward interaction in the spinning Luttinger Model, so obtaining the *g-ology model*. This system is no more solvable; however, a perturbative Renormalization Group (RG) analysis, [12], shows that, in the *repulsive case*, such extra coupling is *marginally irrelevant*, i.e. becomes negligible over large space-time scales. In [13] the necessity of implementing Ward Identities in a RG approach was emphasized, in order to go beyond purely perturbative results, but the analysis was limited to the Luttinger model and no attempt was done to include the effects of nonlinear bands. In [14] it was observed that the correlations of the repulsive g-ology model would qualitatively differ from the Luttinger model ones for the presence of multiplicative *logarithmic corrections*.

A new point of view, that extended previous ideas of Kadanoff, [15], and Luther and Peschel, [16], was proposed by Haldane, [17], and is nowadays known as *Luttinger Liquid Conjecture*. The

idea is to exploit the concept of *universality*, a basic notion in statistical physics saying that the critical properties are largely independent from the details of the model, at least inside a certain class of models. In the present case, as the exponents are non trivial functions of the coupling, universality has a meaning more subtle than usual; it does not mean that the exponents are the same (the exponents *do depend* on the details of the model), but that the exponents and certain thermodynamic quantities verify a set of *universal relations* which are *identical* to the Luttinger model ones. Such relations give an exact determination of physical quantities in terms of a few measurable parameters. The validity of such relations has been checked in certain special solvable spinless fermionic lattice models [17], but a proof of their validity in the Hubbard model (or in the non solvable extended Hubbard model) is an open problem. It should be remarked that, even though the Hubbard model differs from the spinning Luttinger model for irrelevant or marginally irrelevant terms in the Renormalization Group sense (in the weak non half filled band case), this would not imply at all the validity of the same relations as in the Luttinger model; irrelevant terms *do renormalize* the exponents and the thermodynamic quantities.

Starting from the 90's, the methods developed in constructive Quantum Field Theory (QFT) for the analysis of QFT models in  $d = 1 + 1$  [18, 19] were applied to interacting non relativistic spinless fermionic systems in the continuum [20], so establishing the anomalous dimension in a non solvable model, by combining Renormalization Group methods with non perturbative information coming from the *exact solution of the Luttinger model*; the extension to spinning fermions was done in [21]. An important technical advance was achieved in [22, 23], by implementing Ward Identities based on local symmetries with Renormalization Group methods. A well known difficulty in any Wilsonian Renormalization Group approach is that the momentum cut-offs break the local symmetries on which Ward Identities are based; in [22, 23] it was developed a technique allowing to rigorously take into account the extra terms produced in the Ward Identities by the cut-offs, so that interacting non relativistic fermions in  $d = 1$  were constructed *without any use of exact solutions* [23, 24, 25]. The main outcome, with respect to the early Renormalization Group analysis [12], is that the results are exact (the lattice and non linear bands are fully taking into account) and non-perturbative; and physical observables are written in terms of convergent expansions so that they can be computed with arbitrary precision. The complexity of such expansions made however impossible to verify explicitly the universal Luttinger Liquid relations in [26]; they have finally been proven for interacting *spinless* fermions on the lattice (the  $XXZ$  chain and extended versions) in [27, 28] through Ward identities; this fact appears to be related to a non-perturbative version of the Adler-Bardeen theorem of the non renormalization of the anomalies, [29]. In this paper we will extend such ideas to *spinning* fermions in the Hubbard model; as we will see, the extension is rather non trivial and new phenomena take place.

## 1.2 Extended Hubbard Model and Physical Observables

Let  $\beta > 0$  be the *inverse temperature*,  $-\mu$  the *chemical potential* and  $\mathcal{C} = \{ -[L/2], \dots, [(L-1)/2]L \}$  a one dimensional lattice of  $L$  sites. The extended Hubbard model [30] describes fermions hopping on  $\mathcal{C}$  with a short-range density-density interaction; the Hamiltonian plus the chemical potential term is

$$H = -\frac{1}{2} \sum_{\substack{x \in \mathcal{C} \\ s = \pm}} (a_{x,s}^+ a_{x+1,s}^- + a_{x,s}^+ a_{x-1,s}^-) + \mu \sum_{\substack{x \in \mathcal{C} \\ s = \pm 1}} a_{x,s}^+ a_{x,s}^- + \lambda \sum_{\substack{x, y \in \mathcal{C} \\ s, s' = \pm 1}} v(x-y) a_{x,s}^+ a_{x,s}^- a_{y,s'}^+ a_{y,s'}^- \quad (1.1)$$

where  $a_{x,s}^\pm$  are fermionic creation and annihilation operators at site  $x$  with 'spin'  $s$ , and  $v(x)$  is a smooth, even potential such that  $|v(x)| \leq C e^{-\kappa|x|}$  (short range condition); periodic boundary conditions are assumed:  $a_{L+1,s}^\varepsilon = a_{1,s}^\varepsilon$ .

The set-up of the Grand Canonical Ensemble is standard, and we remind it concisely; more details are, for example, in [31]. If  $O_x$  is a monomial in the operators  $a_{x,s}^\varepsilon$ ,  $\varepsilon, s = \pm$ , or in

the density operators  $a_{x,s}^\varepsilon a_{x,s'}^{\varepsilon'}$ ,  $\varepsilon, \varepsilon', s, s' = \pm$ , given  $x_0 \in [0, \beta]$ , define  $\mathbf{x} = (x, x_0)$  and  $O_{\mathbf{x}} := e^{Hx_0} O_x e^{-Hx_0}$  (so that  $x_0$  has the meaning of *imaginary* time); then, given the observables  $O_{\mathbf{x}_1}, \dots, O_{\mathbf{x}_n}$ , their Grand Canonical correlation is

$$\langle O_{\mathbf{x}_1} \cdots O_{\mathbf{x}_n} \rangle_{L,\beta} := \frac{\text{Tr}[e^{-\beta H} \mathbf{T}(O_{\mathbf{x}_1} \cdots O_{\mathbf{x}_n})]}{\text{Tr}[e^{-\beta H}]} \quad (1.2)$$

where  $\mathbf{T}$  is the time order product. Similarly,  $\langle O_{\mathbf{x}_1}; \cdots; O_{\mathbf{x}_n} \rangle_{T;L,\beta}$  denotes the corresponding truncated correlations. We are interested in the correlations in the thermodynamic limit  $L \rightarrow \infty$  and at the critical temperature  $\beta^{-1} = 0$ ; the limit  $L, \beta \rightarrow \infty$  will be indicated by dropping the labels  $L, \beta$ .

Define  $\bar{p}_F \in [0, \pi]$ , the *free Fermi momentum*, and  $\bar{v}_F$ , the *free Fermi velocity*, such that

$$\cos \bar{p}_F = \mu \quad \bar{v}_F = \sin \bar{p}_F .$$

In this paper we have three main assumptions on the parameters:

$$\bar{p}_F \neq 0, \pi/2, \pi, \quad \hat{v}(2\bar{p}_F) > 0, \quad \lambda \geq 0 \quad (1.3)$$

The condition  $\bar{p}_F \neq 0, \pi$  means that the *empty band* and the *filled band* cases are *not* included; the reason of such exclusion is that, if  $\bar{v}_F = 0$ , the scaling of the model would be very different and would depend in a critical way on the interaction. The condition  $\bar{p}_F \neq \pi/2$  excludes the *half-filled band* case; it will have the effect to make the Umklapp interaction (see §2.1) irrelevant (in the RG language). The two other conditions can be loosely called the *repulsive condition* on the interaction; they indeed imply that one of the contribution to the effective interaction (in the RG language) is strictly positive at all scales.

The model is  $SU(2)$  symmetric, as the Hamiltonian is invariant under transformation  $a_{x,s}^\pm \rightarrow \sum_{s'} M_{s,s'} a_{x,s'}^\pm$  with  $M \in SU(2)$ ; and includes the standard and the U-V Hubbard models, corresponding to the interactions  $\lambda v(x-y) = U\delta_{x,y}$  and  $\lambda v(x-y) = U\delta_{x,y} + \frac{1}{2}V\delta_{|x-y|,1}$ , respectively: in the former case the repulsive condition is  $U \geq 0$ .

By definition, the  $T = 0$  *free energy* is

$$E(\lambda) := - \lim_{L, \beta \rightarrow \infty} (L\beta)^{-1} \log \text{Tr}[e^{-\beta H}] , \quad (1.4)$$

and the 2-point *Schwinger function* is

$$S_{2,\beta,L}(\mathbf{x} - \mathbf{y}) := \langle a_{\mathbf{x},+}^- a_{\mathbf{y},+}^+ \rangle_{\beta,L} = \langle a_{\mathbf{x},-}^- a_{\mathbf{y},-}^+ \rangle_{\beta,L} . \quad (1.5)$$

The connection with experimental physics is through the *response functions*, defined as Fourier transforms of the following truncated correlations:

$$\Omega_{\alpha,\beta,L}(\mathbf{x} - \mathbf{y}) := \langle \rho_{\mathbf{x}}^\alpha \rho_{\mathbf{y}}^\alpha \rangle_{T;\beta,L} := \langle \rho_{\mathbf{x}}^\alpha \rho_{\mathbf{y}}^\alpha \rangle_{\beta,L} - \langle \rho_{\mathbf{x}}^\alpha \rangle_{\beta,L} \langle \rho_{\mathbf{y}}^\alpha \rangle_{\beta,L} \quad (1.6)$$

where  $\rho_{\mathbf{x}}^\alpha$  is one of the following densities (see pagg. 54, 55 of [4]):

$$\begin{aligned} \rho_{\mathbf{x}}^C &= \sum_{s=\pm} a_{\mathbf{x},s}^+ a_{\mathbf{x},s}^- && \text{(charge density)} \\ \rho_{\mathbf{x}}^{S_i} &= \sum_{s,s'=\pm} a_{\mathbf{x},s}^+ \sigma_{s,s'}^{(i)} a_{\mathbf{x},s'}^- && \text{(spin densities)} \\ \rho_{\mathbf{x}}^{SC} &= \frac{1}{2} \sum_{\substack{s=\pm \\ \varepsilon=\pm}} s a_{\mathbf{x},s}^\varepsilon a_{\mathbf{x},-s}^\varepsilon && \text{(singlet Cooper density)} \\ \rho_{\mathbf{x}}^{TC_i} &= \frac{1}{2} \sum_{\substack{s,s'=\pm \\ \varepsilon=\pm}} a_{\mathbf{x},s}^\varepsilon \tilde{\sigma}_{s,s'}^{(i)} a_{\mathbf{x}+\mathbf{e},s'}^\varepsilon, \quad \mathbf{e} = (1, 0) && \text{(triplet Cooper densities)} \end{aligned} \quad (1.7)$$

where  $i = 1, 2, 3$  and

$$\begin{aligned}\sigma^{(1)} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \sigma^{(2)} &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \sigma^{(3)} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \tilde{\sigma}^{(1)} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \tilde{\sigma}^{(2)} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \tilde{\sigma}^{(3)} &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

When the interaction is off ( $\lambda = 0$ ), all functions  $\Omega_\alpha(\mathbf{x} - \mathbf{y})$  have power law decay for large  $|\mathbf{x} - \mathbf{y}|$  with the same exponent; as we will see, turning on the interaction removes such degeneracy: some correlations are enhanced by the presence of interaction and others are depressed, so that the exponents become different.

Define the Fourier transform of the response functions as

$$\hat{\Omega}_\alpha(\mathbf{p}) := \lim_{\beta, L \rightarrow \infty} \int_{-\beta/2}^{\beta/2} dx_0 \sum_{x \in \mathcal{C}} e^{i\mathbf{p}\mathbf{x}} \Omega_{\alpha, \beta, L}(\mathbf{x}) \quad (1.8)$$

where  $\mathbf{p} = (p, p_0)$ , with  $p \in \frac{2\pi}{L}\mathcal{C}$  and  $p_0 \in \frac{2\pi}{\beta}\mathbb{Z}$ . The *susceptibility* is given by <sup>1</sup>

$$\kappa := \lim_{p \rightarrow 0} \hat{\Omega}_C(p, 0). \quad (1.9)$$

The paramagnetic part of the current  $J_x$  is defined as

$$J_x = \frac{1}{2i} \sum_{s=\pm} [a_{x+1,s}^+ a_{x,s}^- - a_{x,s}^+ a_{x+1,s}^-] \quad (1.10)$$

while the diamagnetic part is

$$\tau_x = -\frac{1}{2} \sum_{s=\pm} [a_{x,s}^+ a_{x+1,s}^- + a_{x+1,s}^+ a_{x,s}^-] \quad (1.11)$$

The *Drude weight* is defined as

$$D = -\langle \tau_x \rangle - \lim_{p_0 \rightarrow 0} \lim_{p \rightarrow 0} \lim_{\beta, L \rightarrow \infty} \int_{-\beta/2}^{\beta/2} dx_0 \sum_{x \in \Lambda} e^{i\mathbf{p}\mathbf{x}} \langle J_{\mathbf{x}} J_0 \rangle_{T; L, \beta} \equiv \lim_{p_0 \rightarrow 0} \lim_{p \rightarrow 0} \hat{D}(\mathbf{p}) \quad (1.12)$$

where the first term is a constant independent from  $x$ . If one assumes analytic continuation in  $p_0$  around  $p_0 = 0$ , one can compute the conductivity in the linear response approximation by the Kubo formula, that is  $\sigma = \lim_{\omega \rightarrow 0} \lim_{\delta \rightarrow 0} \frac{\hat{D}(-i\omega + \delta, 0)}{-i\omega + \delta}$ . Therefore, a nonvanishing  $D$  indicates infinite conductivity.

The conservation law

$$\frac{\partial \rho_{\mathbf{x}}^C}{\partial x_0} = e^{Hx_0} [H, \rho_x] e^{-Hx_0} = -i \partial_x^{(1)} J_{\mathbf{x}} \equiv -i [J_{x, x_0} - J_{x-1, x_0}], \quad (1.13)$$

where  $\partial_x^{(1)}$  denotes the lattice derivative, implies exact relations, called *Ward identities* (WI), between the Schwinger functions, the density correlations and the *vertex functions*, defined as  $G_{\rho}^{2,1}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \langle \rho_{\mathbf{x}}^{(C)} a_{\mathbf{y}}^- a_{\mathbf{z}}^+ \rangle_T$  and  $G_j^{2,1}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \langle J_{\mathbf{x}} a_{\mathbf{y}}^- a_{\mathbf{z}}^+ \rangle_T$ . Some Ward Identities, which will play an important role in the following, are

$$-ip_0 \hat{G}_{\rho}^{2,1}(\mathbf{k}, \mathbf{k} + \mathbf{p}) - i(1 - e^{-ip}) \hat{G}_j^{2,1}(\mathbf{k}, \mathbf{k} + \mathbf{p}) = \hat{S}_2(\mathbf{k}) - S_2(\mathbf{k} + \mathbf{p}) \quad (1.14)$$

$$-ip_0 \hat{\Omega}_C(\mathbf{p}) - i(1 - e^{-ip}) \hat{\Omega}_{j, \rho}(\mathbf{p}) = 0 \quad (1.15)$$

$$-ip_0 \hat{\Omega}_{\rho, j}(\mathbf{p}) - i(1 - e^{-ip}) \hat{D}(\mathbf{p}) = 0 \quad (1.16)$$

where  $\Omega_{j, \rho}(\mathbf{x}, \mathbf{y}) = \langle \rho_{\mathbf{x}}^C J_{\mathbf{y}} \rangle_{T, \beta, L}$ .

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<sup>1</sup>in a fermion system,  $\kappa = \kappa_c \rho^2$ , where  $\kappa_c$  is the *compressibility* and  $\rho$  the *density*, see e.g. (2.83) of [32]

### 1.3 Anomalous exponents and logarithmic corrections

In the free case,  $\lambda = 0$ ,  $\bar{p}_F$  determines the position of the two singularities of the Fourier transform of  $S_2$ , which are at  $\mathbf{k} = (\pm\bar{p}_F, 0)$ ; whereas  $\bar{v}_F$  is the velocity of the large-distance leading term of  $S_2$ ,

$$S_2(\mathbf{x}) \sim \sum_{\omega=\pm} \frac{e^{-i\omega\bar{p}_F x}}{\bar{v}_F x_0 + i\omega x}.$$

In the following theorem we see that, when the interaction is turned on,  $\lambda > 0$ , the singularities of the Fourier transform of  $S_2$  are moved into new positions,  $\mathbf{k} = (\pm p_F, 0)$ , with  $p_F = \bar{p}_F + O(\lambda)$ . Moreover, the power law decay of  $S_2(\mathbf{x})$  and the response functions is strongly modified: the decay exponent is changed (*anomalous dimension*) and there are logarithmic corrections.

**Theorem 1.1** *If the conditions (1.3) are satisfied, there exist  $\lambda_0 > 0$  such that, if  $0 < \lambda < \lambda_0$ , there exist continuous functions*

$$p_F \equiv p_F(\lambda) = \bar{p}_F + O(\lambda) \quad v_F \equiv v_F(\lambda) = \sin p_F(\lambda) + O(\lambda)$$

(depending also of the other parameters of the model, like  $v(x)$  and  $\mu$ ), such that, setting

$$\begin{aligned} \tilde{\mathbf{x}} &:= (x, v_F x_0), \quad L(\mathbf{x}) = 1 + b\lambda\bar{v}(2\bar{p}_F) \log |\mathbf{x}|, \quad b = 2(\pi \sin \bar{p}_F)^{-1} \\ \bar{\Omega}_0(\mathbf{x}) &:= \frac{x_0^2 - x^2}{x_0^2 + x^2}, \quad \bar{S}_0(\mathbf{x}) := \frac{v_F x_0 \cos p_F - x \sin p_F}{|\tilde{\mathbf{x}}|} \end{aligned} \quad (1.17)$$

the large  $|\mathbf{x}|$  asymptotics of the two-points Schwinger function is

$$S_2(\mathbf{x}) \sim [\bar{S}_0(\mathbf{x}) + R_2(\mathbf{x})] \frac{L(\mathbf{x})^{\zeta_z}}{|\tilde{\mathbf{x}}|^{1+\eta}} \quad (1.18)$$

where  $R_2(\mathbf{x})$  is a continuous function of  $\lambda$  and  $\mathbf{x}$ , such that  $|R_2(\mathbf{x})| \leq C_\vartheta \lambda^{1-\vartheta}$ , for some positive constants  $C_\vartheta$  and  $\vartheta < 1$ . Besides, the large  $|\mathbf{x}|$  asymptotics of the correlations are

$$\begin{aligned} \text{for } \alpha = C, S_i \quad \Omega_\alpha(\mathbf{x}) &\sim \frac{\bar{\Omega}_0(\tilde{\mathbf{x}}) + R_\alpha(\mathbf{x})}{\pi^2 |\tilde{\mathbf{x}}|^2} + \cos[2p_F x] \frac{L(\mathbf{x})^{\zeta_\alpha}}{\pi^2 |\tilde{\mathbf{x}}|^{2X_\alpha}} [1 + \tilde{R}_\alpha(\mathbf{x})] \\ \text{for } \alpha = SC \quad \Omega_\alpha(\mathbf{x}) &\sim - \left[ \bar{\Omega}_0(\tilde{\mathbf{x}}) + \tilde{R}_\alpha(\mathbf{x}) \right] \cos(2p_F x) \frac{L(\mathbf{x})^{\zeta_\alpha}}{\pi^2 |\tilde{\mathbf{x}}|^{2\bar{X}_\alpha}} - \frac{1}{\pi^2} \frac{L(\mathbf{x})^{\zeta_\alpha}}{|\tilde{\mathbf{x}}|^{2X_\alpha}} [1 + R_\alpha(\mathbf{x})] \\ \text{for } \alpha = TC_i \quad \Omega_\alpha(\mathbf{x}) &\sim - \frac{v_F^2}{\pi^2} \frac{L(\mathbf{x})^{\zeta_\alpha}}{|\tilde{\mathbf{x}}|^{2X_\alpha}} [1 + R_\alpha(\mathbf{x})] \end{aligned} \quad (1.19)$$

with the functions  $R_\alpha(\mathbf{x})$  and  $\tilde{R}_\alpha(\mathbf{x})$  having the same properties of  $R_2(\mathbf{x})$ . Moreover, the critical exponents  $\eta$  and  $X_\alpha$ , are continuous functions of  $\lambda$ , while the exponents  $\tilde{\zeta}_{SC}$  and  $\zeta_\alpha$  of the logarithmic corrections could also depend on  $\mathbf{x}$  (we can not exclude it), but satisfy the bounds  $|\tilde{\zeta}_{SC}| \leq C\lambda$  and  $|\zeta_\alpha - \bar{\zeta}_\alpha| \leq C\lambda$ , for a suitable constant  $C$ , with

$$\bar{\zeta}_z = 0, \quad \bar{\zeta}_C = -\frac{3}{2}, \quad \bar{\zeta}_{S_i} = \frac{1}{2}, \quad \bar{\zeta}_{SC} = -\frac{3}{2}, \quad \bar{\zeta}_{TC_i} = \frac{1}{2} \quad (1.20)$$

In the free  $\lambda = 0$  case the response functions decay for large distance with power laws of exponent equal to 2. The interaction partially removes such degeneracy by producing *anomalous exponents* which are (in general) non trivial functions of the coupling (see Theorem 1.2); in particular the response to charge and spin densities are *enhanced* by the interaction, while the response to triplet Cooper densities are depressed. While the presence of non universal exponents

is a common feature with the Luttinger model, the presence of *logarithmic corrections* is a striking difference. Such corrections remove the degeneracy in the response of charge and spin densities: the response to spin density is dominating. Note on the other hand that the exponents of the non oscillating part of charge or spin density correlations are the same as in the free case; also logarithmic corrections are excluded.

Similar formulas have been derived in the physical literature [14] under the *g-ology approximation*, that is replacing the Hubbard model with the continuum g-ology model describing fermions with linear dispersion relation. The existence of anomalous exponents was proved previously in Theorem 1 of [24] (see also [21]), and the absence of logarithmic corrections in the non oscillating part of charge or spin density correlations was also previously proved in [33]. The above theorem improves such results, as it proves for the first time the existence of logarithmic corrections and universal relations.

## 1.4 The Luttinger liquid relations

**Theorem 1.2** *Under the same conditions of Theorem 1.1, there exist continuous functions*

$$K \equiv K(\lambda) = 1 - c\lambda + O(\lambda^{3/2}), \quad \bar{K} \equiv \bar{K}(\lambda) = 1 - c\lambda + O(\lambda^{3/2}) \quad (1.21)$$

with  $c = [\hat{v}(0) - \hat{v}(2p_F)](\pi \sin \bar{p}_F)^{-1}$ , such that the critical exponents satisfy the extended scaling formulas

$$\begin{aligned} 4\eta &= K + K^{-1} - 2, & 2X_C &= 2X_{S_i} = K + 1, \\ 2X_{TC_i} &= 2X_{SC} = K^{-1} + 1, & 2\tilde{X}_{SC} &= K + K^{-1}. \end{aligned} \quad (1.22)$$

Moreover

$$\begin{aligned} \hat{\Omega}_C(\mathbf{p}) &= \frac{\bar{K}}{\pi v} \frac{v^2 p^2}{p_0^2 + v^2 p^2} + A(\mathbf{p}) \\ \hat{D}(\mathbf{p}) &= \frac{v}{\pi} \bar{K} \frac{p_0^2}{p_0^2 + v^2 p^2} + B(\mathbf{p}) \end{aligned} \quad (1.23)$$

with  $A(\mathbf{p})$ ,  $B(\mathbf{p})$  continuous and vanishing at  $\mathbf{p} = 0$ ,  $v = \sin \bar{p}_F + O(\lambda)$  and  $\bar{K} = 1 + O(\lambda)$ ; therefore the Drude weight  $D$  (1.12) and the susceptibility  $\kappa$  (1.9) are  $O(\lambda)$  close to their free values and verify the Luttinger liquid relation

$$v^2 = D/\kappa \quad (1.24)$$

The above Theorem says that, even if the logarithmic corrections alter the power law decay of the spinning Luttinger model, the exponents verify the same *universal relations* (1.22), in agreement with *the Luttinger liquid conjecture* [15, 16, 17]. Such relations say that the knowledge of a single exponents implies the determination of all the others.

The Fourier transform of the density correlation is similar to the free one, the interaction producing a renormalization of the velocity and of the amplitude  $\bar{K}$ . The susceptibility and the Drude weight are *finite*, saying that the system has a metallic behavior (contrary to what happens in the half-filled band case).

Besides the universal relations involving the critical exponents, there is also the *universal relation* (1.24), which relates the susceptibility and the Drude weight to the charge velocity  $v$  appearing in (1.23); this relation was conjectured in [17] (in the spinless case, but the extension to the spinning case is straightforward, see *e.g.* [4]). In the notation of [17],  $v_N = \pi\kappa^{-1}$ ,  $v_J = \frac{D}{\pi}$  so that (1.24) takes the form

$$v_N v_J = v^2 \quad (1.25)$$

The validity of (1.22), (1.24) is a rather remarkable universality property following from the combination of conservation laws of the Hubbard model and Ward Identities coming from the asymptotic gauge invariance of the effective theory. Whether a similar relation holds also for the spin conductivity is an interesting open problem.

## 1.5 Spin Charge separation

**Theorem 1.3** *Under the same condition of Theorem 1.1, the Fourier transform of the 2-point Schwinger function is given by*

$$\hat{S}_2(\mathbf{k} + \mathbf{p}_F^\omega) = Z(\mathbf{k})\hat{S}_{M,\omega}(\mathbf{k})[1 + R(\mathbf{k})], \quad \mathbf{p}_F^\omega = (\omega p_F, 0) \quad (1.26)$$

where

$$|R(\mathbf{k})| \leq C \frac{\lambda^2}{1 + a|\lambda \log |\mathbf{k}||}, \quad a \geq 0, \quad (1.27)$$

$$Z(\mathbf{k}) = L(|\mathbf{k}|^{-1})^{\zeta_z} [1 + R'(\mathbf{k})], \quad |R'(\mathbf{k})| \leq C|\lambda| \quad (1.28)$$

$L(t)$ ,  $t \geq 1$ , is the function defined in Theorem 1.1 and  $\hat{S}_{M,\omega}(\mathbf{k})$  is a function whose Fourier transform is of the form

$$S_{M,\omega}(\mathbf{x}) = \frac{1}{2\pi v_F} \frac{[v_\rho^2 x_0^2 + (x_1/v_F)^2]^{-\eta_\rho/2}}{(v_\rho x_0 + i\omega x_1/v_F)^{1/2} (v_\sigma x_0 + i\omega x_1/v_F)^{1/2}} e^{C+O(1/|\mathbf{x}|)} \quad (1.29)$$

with  $v_{\rho,\sigma} = 1 + O(\lambda)$ ,  $\eta_\rho = O(\lambda^2)$ ,  $v_\rho - v_\sigma = c_v \lambda + O(\lambda^2)$ , with  $c_v \neq 0$ .

The above theorem says that the two point function can be written, up to a logarithmic correction, as the 2-point function of the spinning Luttinger model, a model which shows the phenomenon of *spin-charge separation* (see also [25]). A manifestation of spin charge separation is that the 2-point function is factorized in the product of two functions, similar to Schwinger functions of particles with different velocities. In this sense, the above theorem says that the spin-charge separation occurs approximately also in the Hubbard model, but is valid only at large distances and up to logarithmic corrections. Similar expressions are true also for the density correlations (the explicit formulae are in §5 and are not reported here for brevity). In the spinning Luttinger model  $v_\rho = v$ , where  $v$  is the velocity appearing in (1.23); in the present case we can verify this identity only at the lowest order in  $\lambda$ , and whether this identity holds or not in the Hubbard model is an interesting open problem.

## 1.6 Borel summability

In §2.6 we shall prove the following Theorem.

**Theorem 1.4** *Given  $\delta \in (0, \pi/2)$ , there exists  $\varepsilon \equiv \varepsilon(\delta) > 0$ , such that the free energy, the Schwinger functions and the density correlations are analytic in the set*

$$D_{\varepsilon,\delta} = \{\lambda \in \mathbb{C} : |\lambda| < \varepsilon, |\arg \lambda| < \pi - \delta\} \quad (1.30)$$

and continuous in the closure,  $\bar{D}_{\varepsilon,\delta}$ . Moreover, if  $f(\lambda)$  is one of these functions, there exist three constants  $c_0$ ,  $c_1$ ,  $\kappa$ , and a family of functions  $f_h(\lambda)$ ,  $h \leq 0$ , analytic in the set

$$D_{\varepsilon,\delta}^{(h)} := D_{\varepsilon,\delta} \cup \left\{ \lambda \in \mathbb{C} : |\lambda| < \frac{c_0}{1 + |h|} \right\} \quad (1.31)$$

such that

$$|f_h(\lambda)| \leq c_1 e^{-\kappa|h|}, \quad f(\lambda) = \sum_{h=-\infty}^0 f_h(\lambda) \quad (1.32)$$

By using the Lemma in [19], see pag. 466, this Theorem implies that all the functions satisfy the Watson Theorem, see pag. 192 of [34]. Hence they are Borel summable in the usual meaning.



## 1.7 Contents of the paper

The paper is organized in the following way.

1. In §2.1–2.3 we resume the RG analysis of the extended Hubbard model given in [35, 24]. The fermionic field is decomposed as a sum of fields  $\psi^{(h)}$ ,  $h$  integer and  $\leq 1$ . The field  $\psi^{(1)}$  is associated to the momenta far from the Fermi points, while the fields  $\psi^{(h)}$  with  $h \leq 0$  are supported closer and closer to the two Fermi points, hence are more and more singular in the infrared region. The iterative integration of such fields, accompanied by a *free measure renormalization* (the field strength renormalization), leads to a sequence of *effective potentials*  $V^{(h)}$ , expressed as renormalized expansions in a set of 5 *running couplings*  $\vec{v}_h = (\nu_h, \delta_h, g_{1,h}, g_{2,h}, g_{4,h})$ , whose flow is driven by a recursive relation called *beta function*. The coupling  $\nu_h$  is associated to the only relevant term (in the usual RG language) present in the effective potentials (the others are marginal) and describes the change of the Fermi momentum due to the interaction; in order to control its flow, we change the chemical potential  $\mu$  in  $\mu - \nu$  and we compensate this operation by adding to the interaction a *counterterm*  $\nu \sum_{x,s} a_{x,s}^+ a_{x,s}^-$ . The value of  $\nu$  is then chosen, by an iterative argument, so that  $\nu_h \rightarrow 0$  as  $h \rightarrow -\infty$ ; this will implicitly determine the interacting Fermi momentum. This procedure works since one can prove, under the conditions of Theorem 1.1, the convergence of expansions in the running couplings, by using two crucial technical tools: the determinant bounds for fermionic truncated expectations and the *partial vanishing of the beta function* (see (2.45) below for the definition), which implies the convergence of  $\vec{v}_h$  to  $\vec{v}_{-\infty} = (0, \delta_{-\infty}, 0, g_{2,-\infty}, g_{4,\infty})$ . This limit can be seen as characterizing a point in a set of *fixed points*, depending on 3 parameters, of the RG transformation, suitably scaled; of course, the chosen fixed point depends on all details of the model.
2. In Appendix A a property of the effective couplings flow is proved, implying the convergence of our expansions in the set  $D_{\varepsilon,\delta}$ , defined in (1.30). At this point, as explained in §2.6, simple dimensional arguments allow us to prove Theorem 1.4 and then the Borel summability of the free energy and all the correlation functions.
3. In §2.4–2.5 the analysis is extended to the 2-point function and to the response functions, which are expressed in terms also of renormalizations of the density operators. In particular, from the flow of these renormalizations one obtains the critical exponents and the logarithmic corrections appearing in (1.19). The critical exponents only depend on  $v_F$  and  $\vec{v}_{-\infty}$ , what will play a crucial role in the subsequent analysis. This is apparently not true for the logarithmic corrections; such corrections, absent in the spinless case, are due to the weaker convergence of the effective couplings to their limiting values.
4. The Luttinger liquid relations (1.22) or (1.23) can be checked directly by the expansions at lowest orders, but the complexity of such expansions makes essentially impossible their proof at any order. Similarly, the partial vanishing of the beta function, on which the RG analysis is based, cannot be proven directly from the expansions. Such properties are related to the *asymptotic* validity of certain symmetries, and our strategy consists in the introduction of a suitable *effective model*, for which such symmetries are *exact*, and in showing that certain quantities computed in the effective model coincides, with a proper choice of its parameters, with analogues quantities in the extended Hubbard model. The *effective model* is introduced in §3.1 and is expressed directly in terms of functional integrals with linear dispersion relation; the model has an infrared and an ultraviolet momentum cut-off, and several interactions are present, non local and short ranged (both in space and time). The model can be consider a variation of the *g-ology* models introduced in the physical literature, the main difference being that the cut-offs are on space and time momentum components, what is of advantage for our approach (but makes the model not accessible to bosonization techniques).

5. The non locality of the interaction allows us the removal of the ultraviolet cut-off and a Renormalization Group analysis in the infrared region can be performed (similar to that performed in the Hubbard model), leading to convergent expansions in the effective couplings, see Appendix B. A first use of the effective model is in the proof of the partial vanishing of the beta function of the extended Hubbard model; a proof of this property was already given in [24], but we present here a simplified version of it (the main novelty is that the ultraviolet cut-off in the effective model is removed), see Appendix C. Ward Identities and Schwinger-Dyson equations, with corrections due to the infrared cut-off, can be combined in the absence of back-scattering interactions to get relations implying the partial vanishing of the Beta function of the effective model; from this fact and using the symmetry properties in Appendix B, we can derive the partial vanishing of the beta function of the extended Hubbard model. Note the remarkable fact that a model with no back-scattering interaction and not spin-symmetric is used to prove properties of the Hubbard model, which is spin symmetric and in which the back-scattering interaction is present.
6. If the back-scattering coupling is set equal to zero and both the infrared and ultraviolet cut-offs are removed, the model becomes *exactly solvable*, in the sense that the Schwinger functions verify a set of *closed equations*, obtained combining Ward Identities and Schwinger-Dyson equations, derived in §3.2–3.5. The non locality of the interaction has the effect that the anomalies in the Ward Identities verify the Adler-Bardeen non renormalization property, see [36], [28], so that they can be *exactly computed*; therefore also the critical exponents, which are expressed in terms of such anomalies, can be exactly computed and the analogue of the relations (1.22) for the effective model are obtained.
7. In §3.6 we prove that the limiting values  $\vec{v}_{-\infty}$  of the effective couplings of the effective model with no back-scattering coincide with that of the Hubbard model, if a suitable fine tuning of the *bare* couplings of the effective model is done. Since the critical exponents only depend on  $\vec{v}_{-\infty}$  and have the same functional dependence on it as in the Hubbard model, this implies that also the critical exponents coincide. Therefore we can prove that (1.19) is satisfied for the extended Hubbard model, with the critical exponents verifying the relations (1.22).
8. In §4 the proof of Theorem 1.3 is presented. By using the closed equation of the effective model in the limit of removed cut-offs, we can prove for it the exact Spin-Charge separation. We show that this implies the *approximate* Spin-Charge separation for the extended Hubbard model.
9. Finally, in §5 we compute the Drude weight and the susceptibility and we prove the Luttinger liquid relation (1.24). The proof is based on the fact that these quantities are related to the Fourier transform of some density correlations. However, the bounds obtained in Theorem 1.1 in coordinate space do not allow us to exclude logarithmic singularities. In order to prove their finiteness, we use the Ward Identities of the Hubbard model (1.14), (1.15), (1.16), combined with the information coming from the effective model, keeping in this case also the backward interactions; this works since, even if the effective model is not completely solvable in that case, the Fourier transforms of the density correlations can be still exactly computed from the Ward Identities. We can prove that, by a suitable fine tuning of the couplings of the effective model, the Fourier transforms of the current and density operators coincide up to a constant and a renormalization, whose values are fixed by the Ward Identities (1.14), (1.15), (1.16): this implies (1.24).

## 2 RG Analysis for the Hubbard Model

### 2.1 Functional integral representation

The analysis of the Hubbard model correlations is done by a rigorous implementation of the RG techniques. To begin with, we need a *functional integral representation* of the model, because the RG techniques are optimized for that. We give here a concise description of it; a thorough discussion is in Sec 2 of [24]. The main object to study is the functional  $\mathcal{W}(J, \eta) \equiv \mathcal{W}_{M;L,\beta}(J, \eta)$ , defined by

$$e^{\mathcal{W}(J, \eta)} = \int P(d\psi) e^{-\mathcal{V}(\psi) + \sum_{\alpha} \int d\mathbf{x} J_{\mathbf{x}}^{\alpha} \rho_{\mathbf{x}}^{\alpha} + \sum_s \int d\mathbf{x} [\eta_{\mathbf{x},s}^{+} \psi_{\mathbf{x},s}^{-} + \psi_{\mathbf{x},s}^{+} \eta_{\mathbf{x},s}^{-}]} \quad (2.1)$$

where  $\psi_{\mathbf{x},s}^{\pm}$  and  $\eta_{\mathbf{x},s}^{\pm}$  are Grassmann variables and the fermionic density operators  $\rho_{\mathbf{x}}^{\alpha}$  are defined as in (1.7), with  $\psi_{\mathbf{x},s}^{\pm}$  in place of  $a_{\mathbf{x},s}^{\pm}$ ,  $J_{\mathbf{x}}^{\alpha}$  are commuting variables,  $\int d\mathbf{x}$  is a short form for  $\sum_{x \in \mathcal{C}} \int_{-\beta/2}^{\beta/2} dx_0$ ,  $P(d\psi)$  is a Grassmann Gaussian measure in the field variables  $\psi_{\mathbf{x},s}^{\pm}$  with covariance (the free propagator) given by

$$\begin{aligned} \int P(d\psi) \psi_{\mathbf{x},s}^{\varepsilon} \psi_{\mathbf{y},s'}^{\varepsilon} &= 0, \quad \int P(d\psi) \psi_{\mathbf{x},s}^{-} \psi_{\mathbf{y},-s}^{+} = 0, \\ \int P(d\psi) \psi_{\mathbf{x},s}^{-} \psi_{\mathbf{y},s}^{+} &= \bar{g}_M(\mathbf{x} - \mathbf{y}) := \frac{1}{\beta L} \sum_{\mathbf{k} \in \mathcal{D}_{L,\beta}} \frac{\chi(\gamma^{-M} k_0) e^{ik_0 \delta_M} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}}{-ik_0 + (\cos \bar{p}_F - \cos k)}. \end{aligned} \quad (2.2)$$

In the above formulae,  $\chi(t)$  is a smooth compact support function equal to 1 for  $|t| < 1$  and equal to 0 if  $|t| \geq \gamma$ , for a given *scaling parameter*  $\gamma > 1$ , fixed throughout the paper;  $M$  is a positive integer;  $\mathcal{D}_{L,\beta} := \mathcal{D}_L \times \mathcal{D}_{\beta}$ ,  $\mathcal{D}_L := \frac{2\pi}{L} \mathcal{C}$ ,  $\mathcal{D}_{\beta} := \frac{2\pi}{\beta} (\mathbb{Z} + \frac{1}{2})$ ;

$$\mathcal{V}(\psi) = \lambda \sum_{s,s'=\pm} \int d\mathbf{x} d\mathbf{y} \psi_{\mathbf{x},s}^{+} \psi_{\mathbf{x},s}^{-} v(\mathbf{x} - \mathbf{y}) \psi_{\mathbf{y},s'}^{+} \psi_{\mathbf{y},s'}^{-} \quad (2.3)$$

with  $v(\mathbf{x} - \mathbf{y}) = \delta(x_0 - y_0) v(x - y)$ . Due to the presence of the ultraviolet cut-off  $\gamma^M$ , the Grassmann integral has a finite number of degree of freedom, hence it is well defined. The time shift in (2.2),  $\delta_M := \beta/\sqrt{M}$ , is introduced in order to take correctly into account the discontinuity of  $g(\mathbf{x})$  at  $\mathbf{x} = 0$ : our definition guarantees that, fixed  $L$  and  $\beta$ ,  $\lim_{M \rightarrow \infty} \bar{g}_M(\mathbf{x}) = g(\mathbf{x})$  for  $\mathbf{x} \neq 0$ , while  $\lim_{M \rightarrow \infty} \bar{g}_M(0, 0) = g(0, 0^-)$ , as it is to be for Proposition 2.1 below.

If  $\lambda = 0$ , the Hubbard model correlations can be easily calculated by using (2.2), hence they are singular at momenta  $(\omega \bar{p}_F, 0)$ ,  $\omega = \pm 1$ . Since in the interacting theory,  $\lambda \neq 0$ , the position of the singularity is expected to change by  $O(\lambda)$ , when the first of conditions (1.3) is satisfied, we add to the interaction a *counterterm*

$$\nu \mathcal{N}(\psi) = \nu \sum_{s=\pm} \int d\mathbf{x} \psi_{\mathbf{x},s}^{+} \psi_{\mathbf{x},s}^{-}$$

and, to leave unchanged  $\mathcal{W}(J, \eta)$  in (2.1), we subtract the same term from the free measure, that has then a covariance:

$$g_M(\mathbf{x}) = \frac{1}{\beta L} \sum_{\mathbf{k} \in \mathcal{D}_{L,\beta}} \frac{\chi(\gamma^{-M} k_0) e^{ik_0 \delta_M} e^{-i\mathbf{k}\mathbf{x}}}{-ik_0 + (\cos p_F - \cos k)}, \quad (2.4)$$

where  $p_F$  is the *interacting Fermi momentum* defined such that

$$\cos p_F = \mu - \nu.$$

We introduce the following Grassmann integrals:

$$S_n^{M,\beta,L}(\mathbf{x}_1, s_1, \varepsilon_1; \dots; \mathbf{x}_n, s_n, \varepsilon_n) = \frac{\partial^n}{\partial \eta_{\mathbf{x}_1, s_1}^{-\varepsilon_1} \dots \partial \eta_{\mathbf{x}_n, s_n}^{-\varepsilon_n}} \mathcal{W}(J, \eta) \Big|_{0,0} \quad (2.5)$$

It is well known that such Grassmann integrals, called *Schwinger functions*, can be used to compute the thermodynamical properties of the model with Hamiltonian (1.1). This follows from the following proposition.

**Proposition 2.1** *For any finite  $\beta$  and  $L$ , there exists a complex disc, centered in the origin,  $D_{L,\beta}$ , such that, if  $\lambda \in D_{L,\beta}$ ,*

$$\frac{\text{Tr}[e^{-\beta H} \mathbf{T} \mathbf{a}_{\mathbf{x}_1, s_1}^{\varepsilon_1} \cdots \mathbf{a}_{\mathbf{x}_n, s_n}^{\varepsilon_n}]}{\text{Tr}[e^{-\beta H}]}|_T = \lim_{M \rightarrow \infty} S_n^{M, \beta, L}(\mathbf{x}_1, s_1, \varepsilon_1; \dots; \mathbf{x}_n, s_n, \varepsilon_n); \quad (2.6)$$

*besides both members are analytic in  $\lambda$  in the same disc. A similar statement holds for the density correlations.*

The proof of this theorem can be done exactly as in the spinless case, see [35]. The main point, strictly related with the fact that we are treating a fermionic problem, is that, for  $L$  and  $M$  finite, the l.h.s. of (2.6) is the ratio of the traces of two matrices whose coefficients are entire functions of  $\lambda$ , hence it is the ratio of two entire functions of  $\lambda$ . Then, it may have a singularity only if  $\text{Tr}[e^{-\beta H}]$  vanishes, which certainly does not happen in a neighborhood of  $\lambda = 0$  small enough. On the other hand, it is rather easy to prove that also the r.h.s. of (2.6) is analytic in a small neighborhood of  $\lambda = 0$  and that its Taylor coefficients coincide with those of the l.h.s.. This follows from the fact that the UV singularity of the free propagator is very mild and can be controlled with a trivial resummation, in the RG expansion, of the tadpole terms (see pag. 1383 of [35]). In this resummation, the only important thing to check is that  $\lim_{M \rightarrow \infty} g_M(0, 0) = g(0, 0^-)$  (otherwise the perturbative expansions of the two sides of (2.6) would not coincide).

The RG analysis will allow to prove that the analyticity domain is indeed of the form  $D_{\beta, L} = \{ \lambda, |\lambda| \leq c\varepsilon_0 \min\{(\log \beta)^{-1}, (\log L)^{-1}\} \} \cup \{ |\lambda| \leq \varepsilon_0, |\arg \lambda| < \frac{\pi}{2} + \delta \}$ , with  $c, \varepsilon_0 > 0$ ,  $0 < \delta < \pi/2$  independent of  $\beta$  and  $L$ .

## 2.2 Multiscale analysis for the effective potential

We will briefly recall here the RG analysis for interacting fermionic systems on the lattice as developed in [35] and [24] in the spinless and the spinning case, respectively. Note that the proof of many technical points do not depend on the spin, hence we shall refer to [35] for the corresponding details.

Let  $\mathbb{T}$  be the one dimensional torus,  $\|k - k'\|_{\mathbb{T}}$  the usual distance between  $k$  and  $k'$  in  $\mathbb{T}$  and  $\|k\|_{\mathbb{T}} = \|k - 0\|_{\mathbb{T}}$ . We introduce a positive function  $\chi(\mathbf{k}') \in C^\infty(\mathbb{T} \times \mathbb{R})$ ,  $\mathbf{k}' = (k_0, k')$ , such that  $\chi(\mathbf{k}') = \chi(-\mathbf{k}') = 1$  if  $|\mathbf{k}'| \leq t_0 = a_0 v_F / \gamma$  and  $= 0$  if  $|\mathbf{k}'| > a_0 v_F$ , where  $v_F = \sin p_F$ ,  $a_0 = \min\{\frac{p_F}{2}, \frac{\pi - p_F}{2}\}$  and  $|\mathbf{k}'| = \sqrt{k_0^2 + v_F^2 \|k'\|_{\mathbb{T}}^2}$ . The above definition is such that the supports of  $\chi(k - p_F, k_0)$  and  $\chi(k + p_F, k_0)$  are disjoint and the  $C^\infty$  function on  $\mathbb{T} \times \mathbb{R}$

$$\hat{f}_1(\mathbf{k}) := 1 - \chi(k - p_F, k_0) - \chi(k + p_F, k_0) \quad (2.7)$$

is equal to 0, if  $v_F^2 \| [k] - p_F \|^2_{\mathbb{T}} + k_0^2 < t_0^2$ . We define also, for any integer  $h \leq 0$ ,

$$f_h(\mathbf{k}') = \chi(\gamma^{-h} \mathbf{k}') - \chi(\gamma^{-h+1} \mathbf{k}') \quad (2.8)$$

which has support  $t_0 \gamma^{h-1} \leq |\mathbf{k}'| \leq t_0 \gamma^{h+1}$  and equals 1 at  $|\mathbf{k}'| = t_0 \gamma^h$ ; then

$$\chi(\mathbf{k}') = \sum_{h=h_{L,\beta}}^0 f_h(\mathbf{k}') \quad (2.9)$$

where

$$h_{L,\beta} := \min \{ h : t_0 \gamma^{h+1} > |\mathbf{k}_m| \} \quad \text{for } \mathbf{k}_m = (\pi/\beta, \pi/L). \quad (2.10)$$

For  $h \leq 0$  we also define

$$\hat{f}_h(\mathbf{k}) = f_h(k - p_F, k_0) + f_h(k + p_F, k_0) \quad (2.11)$$

(for  $h = 1$  the definition is (2.7)). This definition implies that, if  $h \leq 0$ , the support of  $\hat{f}_h(\mathbf{k})$  is the union of two disjoint sets,  $A_h^+$  and  $A_h^-$ . In  $A_h^+$ ,  $k$  is strictly positive and  $\|k - p_F\|_{\mathbb{T}} \leq t_0 \gamma^h \leq t_0$ , while, in  $A_h^-$ ,  $k$  is strictly negative and  $\|k + p_F\|_{\mathbb{T}} \leq t_0 \gamma^h$ . The label  $h$  is called the *scale* or *frequency* label. Note that

$$1 = \sum_{h=h_{L,\beta}}^1 \hat{f}_h(\mathbf{k}) \quad (2.12)$$

and that, since  $p_F$  is not uniquely defined at finite volume (we are interested only to the zero temperature limit), then we can redefine it as  $(2\pi/L)(n_F + 1/2)$ , with  $n_F = [Lp_F/(2\pi)]$ . Hence, if  $\mathcal{D}'_L \equiv \frac{2\pi}{L}(\mathcal{C} + \frac{1}{2})$  and  $\mathcal{D}'_{L,\beta} = \mathcal{D}'_L \times \mathcal{D}_\beta$ , we can write:

$$g(\mathbf{x} - \mathbf{y}) = g^{(1)}(\mathbf{x} - \mathbf{y}) + \sum_{\omega=\pm} \sum_{h=h_{L,\beta}}^0 e^{-i\omega p_F(x-y)} g_\omega^{(h)}(\mathbf{x} - \mathbf{y}) \quad (2.13)$$

where

$$g^{(1)}(\mathbf{x} - \mathbf{y}) = \frac{1}{\beta L} \sum_{\mathbf{k} \in \mathcal{D}_{L,\beta}} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{\hat{f}_1(\mathbf{k})}{-ik_0 + (\cos p_F - \cos k)} \quad (2.14)$$

$$g_\omega^{(h)}(\mathbf{x} - \mathbf{y}) = \frac{1}{\beta L} \sum_{\mathbf{k}' \in \mathcal{D}'_{L,\beta}} e^{-i\mathbf{k}'(\mathbf{x}-\mathbf{y})} \frac{f_h(\mathbf{k}')}{-ik_0 + E_\omega(k')} \quad (2.15)$$

and

$$E_\omega(k') = \omega v_F \sin k' + \cos p_F (1 - \cos k') \quad (2.16)$$

Notice we have dropped the important phase factor  $e^{ik_0 \delta_M}$  from  $g$ , for it plays no explicit role in the following analysis, since the limit  $N \rightarrow \infty$  is taken before the limits  $L \rightarrow \infty$  and  $\beta \rightarrow \infty$ . As consequence of fundamental properties of the Grassmann Gaussian integration, the decomposition of the covariance (2.13) implies a decomposition of the field

$$\psi_{\mathbf{x},s}^\varepsilon = \psi_{\mathbf{x},s}^{\varepsilon,(1)} + \sum_{\omega=\pm} \sum_{h=h_{L,\beta}}^0 e^{i\omega p_F \varepsilon \mathbf{x}} \psi_{\mathbf{x},\omega,s}^{\varepsilon,(h)} \quad (2.17)$$

where fields with different scale labels or different label  $\omega$  are independent, and the covariance of  $\psi^{(1)}$  is  $g^{(1)}$ , while the covariance of  $\psi_\omega^{(h)}$  is  $g_\omega^{(h)}$ . Basically the label  $\omega$  refers to either two branches of the dispersion relation.

Let us now describe the perturbative expansion of the functional  $\mathcal{W}(J, \eta)$  defined in (2.1); for simplicity we shall consider only the case  $\eta = 0$ . We can write:

$$\begin{aligned} e^{\mathcal{W}(J,0)} &= \int P(d\psi^{\leq 0}) \int P(d\psi^{(1)}) e^{-\mathcal{V}(\psi) - \nu \mathcal{N}(\psi) + \sum_\alpha \int d\mathbf{x} J_\alpha^{(\alpha)} \rho_\alpha^{(\alpha)}} = \\ &= e^{-L\beta E_0} \int P(d\psi^{\leq 0}) e^{-\mathcal{V}^{(0)}(\psi^{\leq 0}) + \mathcal{B}^{(0)}(\psi^{\leq 0}, J)} \end{aligned} \quad (2.18)$$

where, if we put  $\underline{\mathbf{x}} = (\mathbf{x}_1, \dots, \mathbf{x}_{2n})$ ,  $\underline{\omega} = (\omega_1, \dots, \omega_{2n})$  and  $\psi_{\underline{\mathbf{x}}, \underline{\omega}} = \prod_{i=1}^n \psi_{\mathbf{x}_i, \omega_i}^+ \prod_{i=n+1}^{2n} \psi_{\mathbf{x}_i, \omega_i}^-$ , the *effective potential*  $\mathcal{V}^{(0)}(\psi)$  can be represented as

$$\mathcal{V}^{(0)}(\psi) = \sum_{n \geq 1} \sum_{\underline{\omega}} \int d\underline{\mathbf{x}} W_{\underline{\omega}, 2n}^{(0)}(\underline{\mathbf{x}}) \psi_{\underline{\mathbf{x}}, \underline{\omega}} \quad (2.19)$$

the kernels  $W_{\underline{\omega}, 2n}^{(0)}(\underline{\mathbf{x}})$  being analytic functions of  $\lambda$  and  $\nu$  near the origin; if  $|\nu| \leq C|\lambda|$  and we put  $\underline{\mathbf{k}} = (\mathbf{k}_1, \dots, \mathbf{k}_{2n-1})$ , their Fourier transforms satisfy, for any  $n \geq 1$ , the bounds, see §2.4 of [35],

$$|\widehat{W}_{\underline{\omega}, 2n}^{(0)}(\underline{\mathbf{k}})| \leq C^n |\lambda|^{\max\{1, n-1\}} \quad (2.20)$$

A similar representation can be written for the functional  $\mathcal{B}^{(0)}(\psi^{\leq 0}, J)$ , containing all terms which are at least of order one in the external fields, including those which are independent on  $\psi^{\leq 0}$ .

The integration of the scales  $h \leq 0$  is done iteratively in the following way. Suppose that we have integrated the scale  $0, -1, -2, \dots, j$ , obtaining

$$e^{\mathcal{W}(J, 0)} = e^{-L\beta E_j} \int P_{Z_j, C_j}(d\psi^{\leq j}) e^{-\mathcal{V}^{(j)}(\sqrt{Z_j}\psi^{\leq j}) + \mathcal{B}^{(j)}(\sqrt{Z_j}\psi^{\leq j}, J)} \quad (2.21)$$

where, if we put  $C_j(\mathbf{k}')^{-1} = \sum_{h=h_{L, \beta}}^j f_h(\mathbf{k}')$ ,  $P_{Z_j, C_j}$  is the Grassmann integration with propagator

$$\frac{1}{Z_j} g_{\omega}^{(\leq j)}(\mathbf{x} - \mathbf{y}) = \frac{1}{Z_j} \frac{1}{\beta L} \sum_{\mathbf{k} \in \mathcal{D}_{L, \beta}'} e^{-i\mathbf{k}(\mathbf{x} - \mathbf{y})} \frac{C_j^{-1}(\mathbf{k})}{-ik_0 + E_{\omega}(k')} \quad (2.22)$$

$\mathcal{V}^{(j)}(\psi)$  is of the form

$$\mathcal{V}^{(j)}(\psi) = \sum_{n \geq 1} \sum_{\underline{\omega}} \int d\underline{\mathbf{x}} W_{\underline{\omega}, 2n}^{(j)}(\underline{\mathbf{x}}) \psi_{\underline{\mathbf{x}}, \underline{\omega}} \quad (2.23)$$

and  $\mathcal{B}^{(j)}(\psi^{\leq j}, J)$  contains all terms which are at least of order one in the external fields, including those which are independent on  $\psi^{\leq j}$ . For  $j = 0$ ,  $Z_0 = 1$  and the functional  $\mathcal{V}^{(0)}$  and  $\mathcal{B}^{(0)}$  are exactly those appearing in (2.18).

First of all, we define a localization operator in the following way:

$$\begin{aligned} \mathcal{L}\mathcal{V}^{(j)}(\sqrt{Z_j}\psi) &= \gamma^j n_j F_{\nu}(\sqrt{Z_j}\psi) + a_j F_{\alpha}(\sqrt{Z_j}\psi) + z_j F_z(\sqrt{Z_j}\psi) \\ &\quad + l_{1,j} F_1(\psi) + l_{2,j} F_2(\sqrt{Z_j}\psi) + l_{4,j} F_4(\sqrt{Z_j}\psi) \end{aligned} \quad (2.24)$$

where

$$\begin{aligned} F_{\nu} &= \sum_{\omega, s} \int d\mathbf{x} \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, s}^- , & F_1 &= \frac{1}{2} \sum_{\omega, s, s'} \int d\mathbf{x} \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, -\omega, s}^- \psi_{\mathbf{x}, -\omega, s'}^+ \psi_{\mathbf{x}, \omega, s'}^- \\ F_{\alpha} &= \sum_{\omega, s} \int d\mathbf{x} \psi_{\mathbf{x}, \omega, s}^+ \mathcal{D} \psi_{\mathbf{x}, \omega, s}^- , & F_2 &= \frac{1}{2} \sum_{\omega, s, s'} \int d\mathbf{x} \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, s}^- \psi_{\mathbf{x}, -\omega, s'}^+ \psi_{\mathbf{x}, -\omega, s'}^- \\ F_z &= \sum_{\omega, s} \int d\mathbf{x} \psi_{\mathbf{x}, \omega, s}^+ \partial_0 \psi_{\mathbf{x}, \omega, s}^- , & F_4 &= \frac{1}{2} \sum_{\omega, s} \int d\mathbf{x} \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, s}^- \psi_{\mathbf{x}, \omega, -s}^+ \psi_{\mathbf{x}, \omega, -s}^- \end{aligned} \quad (2.25)$$

and  $\mathcal{D}\psi_{\mathbf{x}, \omega, s} = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x}} E_{\omega}(k) \psi_{\mathbf{k}, \omega, s}^+$  (see definition (2.16)). Note that

$$l_{4,0} = 2\lambda \hat{v}(0) + O(\lambda^2) \quad l_{2,0} = 2\lambda \hat{v}(0) + O(\lambda^2) \quad l_{1,0} = 2\lambda \hat{v}(2p_F) + O(\lambda^2) \quad (2.26)$$

and in writing (2.24) the  $SU(2)$  spin symmetry has been used. In the case of local interactions,  $\hat{v}(p) = 1$ .  $F_1$  in (2.25) is called *backward interaction* while  $F_2, F_4$  are the *forward interactions*; the *umklapp interaction*, defined analogously as (B.1) below, is not present in  $\mathcal{L}\mathcal{V}^{(j)}$ , as well as other terms quadratic in the fields. The reason is that the condition  $\bar{p}_F \neq 0, \frac{\pi}{2}, \pi$  says that such terms are vanishing for  $j$  smaller than a suitable constant (depending on  $\bar{p}_F$  and  $\lambda$ ), because they cannot satisfy the conservation of the momentum, so there is no need to localize them (more details are in [24]).

Moreover, the local marginal operators associated with the densities (1.7) are defined in the following way:

$$\mathcal{LB}^{(j)}(\sqrt{Z_j}\psi, J) = \int d\mathbf{x} J_{\mathbf{x}}^{(\alpha)} \left[ \sum_{\alpha \neq TC_i} Z_j^{(1,\alpha)} O_{\mathbf{x}}^{(1,\alpha)}(\psi) + \sum_{\alpha} Z_j^{(2,\alpha)} O_{\mathbf{x}}^{(2,\alpha)}(\psi) \right] \quad (2.27)$$

where  $O^{(1,\alpha)}$  are the *small momentum transfer* correlations, (see pag. 231 of [12])

$$\begin{aligned} O_{\mathbf{x}}^{(1,C)} &= \sum_{\omega, s} \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, s}^- \\ O_{\mathbf{x}}^{(1, S_i)} &= \sum_{\omega, s, s'} \psi_{\mathbf{x}, \omega, s}^+ \sigma_{s, s'}^{(i)} \psi_{\mathbf{x}, \omega, s'}^- \\ O_{\mathbf{x}}^{(1, SC)} &= \sum_{\varepsilon, \omega, s} s e^{2i\varepsilon\omega p_F x} \psi_{\mathbf{x}, \omega, s}^{\varepsilon} \psi_{\mathbf{x}, \omega, -s}^{\varepsilon} \end{aligned} \quad (2.28)$$

while  $O^{(2,\alpha)}$  are the *large momentum transfer* correlations, (see pag. 221 of [12])

$$\begin{aligned} O_{\mathbf{x}}^{(2,C)} &= \sum_{\omega, s} e^{2i\omega p_F x} \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, -\omega, s}^- \\ O_{\mathbf{x}}^{(2, S_i)} &= \sum_{\omega, s, s'} e^{2i\omega p_F x} \psi_{\mathbf{x}, \omega, s}^+ \sigma_{s, s'}^{(i)} \psi_{\mathbf{x}, -\omega, s'}^- \\ O_{\mathbf{x}}^{(2, SC)} &= \sum_{\varepsilon, \omega, s} s \psi_{\mathbf{x}, \omega, s}^{\varepsilon} \psi_{\mathbf{x}, -\omega, -s}^{\varepsilon} \\ O_{\mathbf{x}}^{(2, TC_i)} &= \sum_{\varepsilon, \omega, s, s'} e^{-i\varepsilon\omega p_F} \psi_{\mathbf{x}, \omega, s}^{\varepsilon} \tilde{\sigma}_{s, s'}^{(i)} \psi_{\mathbf{x}, -\omega, s'}^{\varepsilon} \end{aligned} \quad (2.29)$$

These definitions are such that the difference between  $-\mathcal{V}^{(j)} + \mathcal{B}^{(j)}$  and  $-\mathcal{LV}^{(j)} + \mathcal{LB}^{(j)}$  is made of irrelevant terms.

Note that the factor  $e^{-i\varepsilon\omega p_F}$  in the definition of  $O_{\mathbf{x}}^{(2, TC_i)}$  comes from the fact that the two  $a^{\varepsilon}$  operators in the definition (1.7) of the triplet Cooper density are located in two different lattice sites (otherwise the density would vanish). Moreover, there is no local operator  $O_{\mathbf{x}}^{(1, TC_i)}$  because  $\sum_{s, s'} \psi_{\mathbf{x}, \omega, s}^{\varepsilon} \tilde{\sigma}_{s, s'}^{(i)} \psi_{\mathbf{x}, \omega, s'}^{\varepsilon} \equiv 0$  by anticommutation of the fermion fields.

We then renormalize the integration measure, by moving to it part of the quadratic terms in the r.h.s. of (C.5), that is  $-z_j(\beta L)^{-1} \sum_{\omega, s} \sum_{\mathbf{k}} [-ik_0 + E_{\omega}(k)] \psi_{\mathbf{k}, \omega, s}^+ \psi_{\mathbf{k}, \omega, s}^-$ ; equation (2.21) takes the form:

$$e^{\mathcal{W}(J, 0)} = e^{-L\beta(E_j + t_j)} \int P_{\tilde{Z}_{j-1}, C_j}(d\psi^{\leq j}) e^{-\tilde{\mathcal{V}}^{(j)}(\sqrt{Z_j}\psi^{\leq j}) + \mathcal{B}^{(j)}(\sqrt{Z_j}\psi^{\leq j}, J, \tilde{J})} \quad (2.30)$$

where  $\tilde{\mathcal{V}}^{(j)}$  is the remaining part of the effective interaction,  $P_{\tilde{Z}_{j-1}, C_j}(d\psi^{\leq j})$  is the measure whose propagator is obtained by substituting in (2.22)  $Z_j$  with

$$\tilde{Z}_{j-1}(\mathbf{k}) = Z_j[1 + z_j C_j(\mathbf{k})^{-1}] \quad (2.31)$$

and  $t_j$  is a constant coming from the normalization of the measure. It is easy to see that we can decompose the fermion field as  $\psi^{\leq j} = \psi^{\leq j-1} + \psi^{(j)}$ , so that

$$P_{\tilde{Z}_{j-1}, C_j}(d\psi^{\leq j}) = P_{Z_{j-1}, C_{j-1}}(d\psi^{\leq j-1}) P_{Z_{j-1}, \tilde{f}_j^{-1}}(d\psi^{(j)}) \quad (2.32)$$

where  $\tilde{f}_j(\mathbf{k})$  (see eq. (2.90) of [35]) has the same support and scaling properties as  $f_j(\mathbf{k})$ . Hence, if we make the field rescaling  $\psi \rightarrow [\sqrt{Z_{j-1}}/\sqrt{Z_j}]\psi$  and call  $\hat{\mathcal{V}}^{(j)}(\sqrt{Z_{j-1}}\psi^{\leq j})$  the new effective

potential, we can write the integral in the r.h.s. of (2.30) in the form

$$\int P_{Z_{j-1}, C_{j-1}}(d\psi^{(\leq j-1)}) \int P_{Z_{j-1}, \tilde{f}_j^{-1}}(d\psi^{(j)}) e^{-\hat{\mathcal{V}}^{(j)}(\sqrt{Z_{j-1}}\psi^{(\leq j)}) + \hat{\mathcal{B}}^{(j)}(\sqrt{Z_{j-1}}\psi^{(\leq j)}, J, \tilde{J})}$$

By performing the integration over  $\psi^{(j)}$ , we finally get (2.21), with  $j-1$  in place of  $j$ . In order to analyze the result of this iterative procedure, we note that  $\mathcal{L}\hat{\mathcal{V}}^{(j)}(\psi)$  can be written as

$$\mathcal{L}\hat{\mathcal{V}}^{(j)}(\psi) = \gamma^j \nu_j F_\nu(\psi) + \delta_j F_\alpha(\psi) + g_{1,j} F_1(\psi) + g_{2,j} F_2(\psi) + g_{4,j} F_4(\psi) \quad (2.33)$$

where  $\nu_j = (\sqrt{Z_j}/\sqrt{Z_{j-1}})n_j$ ,  $\delta_j = (\sqrt{Z_j}/\sqrt{Z_{j-1}})(a_j - z_j)$  and  $g_{i,j} = (\sqrt{Z_j}/\sqrt{Z_{j-1}})^2 l_{i,j}$ ,  $i = 1, 2, 4$ , are called the *running coupling constants* (r.c.c.) on scale  $j$ .

In Theorem (3.12) of [35] it is proved that the kernels of  $\hat{\mathcal{V}}^{(j)}$  and  $\hat{\mathcal{B}}^{(j)}$  are *analytic* as functions of the r.c.c., provided that they are small enough. One has then to analyze the flow of the r.c.c. (the *beta function*) as  $j \rightarrow -\infty$ . We shall now summarize the results, following §4 and §5 of [24] with some improvement.

### 2.3 The flow of the running coupling constants

Define vector notations for the r.c.c.,

$$\vec{v}_h \equiv (v_{1,h}, v_{2,h}, v_{4,h}, v_{\delta,h}, v_{\nu,h}) = (g_{1,h}, g_{2,h}, g_{4,h}, \delta_h, \nu_h) \equiv (\vec{g}_h, \delta_h, \nu_h). \quad (2.34)$$

The r.c.c. satisfy a set of recursive equations, which can be written in the form

$$v_{\alpha,j-1} = A_\alpha v_{\alpha,j} + \hat{\beta}_\alpha^{(j)}(\vec{v}_j; \dots, \vec{v}_0; \lambda, \nu) \quad (2.35)$$

with  $A_\nu = \gamma$ ,  $A_\alpha = 1$  for  $\alpha \neq \nu$ . These equations have been already analyzed in [24], where it has been proved that, if  $\lambda$  is real positive and small enough, then it is possible to choose  $\nu$  so that, fixed  $\vartheta < 1$ ,  $|\nu_h| \leq C\lambda\gamma^{\vartheta h}$ ,  $\forall h \leq 0$ , and  $0 < g_{1,h} < \lambda(1 + \bar{a}\lambda|h|)^{-1}$ , for some  $\bar{a} > 0$ , while the other r.c.c. stay bounded by  $C\lambda$  and converge for  $h \rightarrow -\infty$ . In this paper, in order to proof Borel summability of perturbation theory, we extend the proof to complex values of  $\lambda$ , restricted to the set  $D_{\varepsilon,\delta}$  defined in (1.30); this implies that we need an analysis a bit more precise of the flow equations (2.35).

To begin with, we put  $\nu_1 \equiv \nu$  and we suppose that the sequence  $\{\nu_h\}_{h \leq 1}$  are known functions of  $\lambda$ , analytic in  $D_{\varepsilon,\delta}$ , such that

$$|\nu_h| \leq C|\lambda|\gamma^{\vartheta h}, \quad h \leq 1 \quad (2.36)$$

and study the flow equations of the other variables. The idea is that this restricted flow has properties such that, by a fixed point argument, the sequence  $\{\nu_h\}_{h \leq 1}$ , satisfying the last equation of (2.35), can be uniquely determined. Since this point can be treated in the same way as in spinless case (see §4.3 of [35]), we shall give for granted this result. Hence, from now on, we shall define  $\mathbf{v}_j = (g_{1,j}, g_{2,j}, g_{4,j}, \delta_j)$  and we shall consider the restriction of (2.35) to  $\mathbf{v}_j$ .

The next step is to extract from the functions  $\hat{\beta}_\alpha^{(j)}$  the leading terms for  $j \rightarrow -\infty$ . Observe that the propagator  $\tilde{g}_\omega^{(j)}$  of the single scale measure  $P_{Z_{j-1}, \tilde{f}_j^{-1}}$ , can be decomposed as

$$\tilde{g}_\omega^{(j)}(\mathbf{x}) = \frac{1}{Z_j} g_{D,\omega}^{(j)}(\mathbf{x}) + r_\omega^{(j)}(\mathbf{x}) \quad (2.37)$$

where  $g_{D,\omega}^{(j)}$  is the *Dirac propagator* (with cutoff) and describes the leading asymptotic behavior

$$g_{D,\omega}^{(j)}(\mathbf{x}) := \frac{1}{\beta L} \sum_{\mathbf{k} \in \mathcal{D}_{L,\beta}} e^{-i\mathbf{k}\mathbf{x}} \frac{\tilde{f}_j(\mathbf{k})}{-ik_0 + \omega v_F k}, \quad (2.38)$$



while the *remainder*  $r_\omega^{(j)}$  satisfies, for any  $q > 0$  and  $0 < \vartheta < 1$ , the bound

$$|r_\omega^{(j)}(\mathbf{x})| \leq \frac{\gamma^{(1+\vartheta)j}}{Z_j} \frac{C_{q,\vartheta}}{1 + (\gamma^j |\mathbf{x}|)^q} . \quad (2.39)$$

Let us now call  $Z_{D,j}$  the values of  $Z_j$  one would obtain by substituting  $\mathcal{V}^{(0)}$  with  $\mathcal{LV}^{(0)}$  in (2.18) and by using for the single scale integrations the propagator (2.37) with  $r_\omega^{(i)}(\mathbf{x}) \equiv 0$  for any  $i \geq j$ . It can be proved by an inductive argument, see §4 of [35], that, if all the r.c.c. stay of order  $\lambda$ ,

$$\left| \frac{Z_j}{Z_{j-1}} - \frac{Z_{D,j}}{Z_{D,j-1}} \right| \leq C \varepsilon_j^2 \gamma^{\vartheta j} \quad (2.40)$$

where

$$\varepsilon_j = \max\{|\lambda|, \max_{0 \leq h \geq j} |\vec{g}_h|, \max_{0 \leq h \geq j} |\delta_h|\} .$$

It is then convenient to decompose the functions  $\hat{\beta}_\alpha^{(j)}$  as

$$\hat{\beta}_\alpha^{(j)}(\vec{v}_j; \dots, \vec{v}_0; \lambda, \nu) = \beta_\alpha^{(j)}(\mathbf{v}_j, \dots, \mathbf{v}_0) + \bar{\beta}_\alpha^{(j)}(\vec{v}_j; \dots, \vec{v}_0; \lambda) \quad (2.41)$$

where  $\beta_\alpha^{(j)}(\mathbf{v}_j, \dots, \mathbf{v}_0)$  is given by the sum of all trees containing only endpoints with r.c.c.  $\delta_h, \vec{g}_h$ ,  $0 \geq h \geq j$ , modified so that the propagators  $g_\omega^{(h)}$  and the wave function renormalizations  $Z_h$ ,  $0 \geq h \geq j$ , are replaced by  $g_{D,\omega}^{(h)}$  and  $Z_{D,h}$ ;  $\bar{\beta}_\alpha^{(j)}$  contains the correction terms together with the remainder of the expansion.

**Lemma 2.2**

$$|\bar{\beta}_\alpha^{(j)}(\vec{v}_j; \dots, \vec{v}_0; \lambda)| \leq \begin{cases} C \varepsilon_j^2 \gamma^{\vartheta j} & \text{if } \alpha \neq \delta \\ (\bar{c} \varepsilon_0 + C \varepsilon_j^2) \gamma^{\vartheta j} & \text{if } \alpha = \delta \end{cases} \quad (2.42)$$

As showed in [24], this lemma is basically a consequence of (2.40) and (2.36). Therefore the leading term in (2.41) is  $\beta_\alpha^{(j)}$ , that we further decompose as

$$\beta_\alpha^{(j)}(\mathbf{v}_j, \dots, \mathbf{v}_0) = \tilde{\beta}_\alpha^{(j)}(\mathbf{v}_j) + r_{\alpha,j}(\mathbf{v}_j, \dots, \mathbf{v}_0) \quad (2.43)$$

where  $\tilde{\beta}_\alpha^{(j)}(\mathbf{v}) = \beta_\alpha^{(j)}(\mathbf{v}, \dots, \mathbf{v})$ . We can write:

$$\tilde{\beta}_\alpha^{(j)}(\mathbf{v}_j) = \sum_{i=0,1} b_{\alpha,i}^{(j)}(\mathbf{v}_j) + b_{\alpha,\geq 2}^{(j)}(\mathbf{v}_j) \quad (2.44)$$

where  $b_{\alpha,i}^{(j)}(\mathbf{v}_j)$  is the contribution of order  $i$  in  $g_{1,j}$ , while  $b_{\alpha,\geq 2}^{(j)}(\mathbf{v}_j)$  is the contributions of all trees with at least two endpoints of type  $g_1$ . The crucial property is the following lemma.

**Lemma 2.3 (partial vanishing of the beta function)**

$$|b_{\alpha,i}^{(j)}(\mathbf{v}_j)| \leq C \varepsilon_j^2 \gamma^{\vartheta j} , \quad i = 0, 1 \quad (2.45)$$

The above property was proven in §5.3 of [24], extending the proof for the spinless case in [37, 23], and it will be reviewed in App. C. Now, let us extract from  $\tilde{\beta}_\alpha^{(j)}(\mathbf{v}_j)$  the second order contributions, which all belong to  $b_{\alpha,\geq 2}^{(j)}(\mathbf{v}_j)$ ; we get:

$$\tilde{\beta}_\alpha^{(j)}(\mathbf{v}_j) = -a_\alpha g_{1,j}^2 + \sum_{i=0,1} b_{\alpha,i}^{(j)}(\mathbf{v}_j) + \tilde{r}_{\alpha,j}(\mathbf{v}_j) \quad (2.46)$$

with  $a_1 = a > 0$ ,  $a_2 = a/2$ ,  $a_4 = a_\delta = 0$ , and, for some  $b_1 > 0$ ,

$$|\tilde{r}_{\alpha,j}(\mathbf{v}_j)| \leq b_1 \varepsilon_j |g_{1,j}|^2. \quad (2.47)$$

In the limit  $L, \beta = \infty$ , if  $g_{D,\omega}^{(\geq h)} \equiv \sum_{j=h}^0 g_{D,\omega}^{(j)}$ ,

$$a = 2 \lim_{h \rightarrow -\infty} \frac{1}{|h|} \int \frac{dk}{(2\pi)^2} \hat{g}_{D,+}^{(\geq h)}(\mathbf{k}) \hat{g}_{D,-}^{(\geq h)}(\mathbf{k}) = \frac{\log \gamma}{\pi v_F} \quad (2.48)$$

Let us now analyze in more detail the functions  $r_\alpha^{(j)}(\mathbf{v}_j, \dots, \mathbf{v}_0)$ , which appear in (2.43). If we define, for  $j' \geq j+1$ ,

$$D_\alpha^{(j,j')}(\mathbf{v}_j, \dots, \mathbf{v}_0) = \beta_\alpha^{(j)}(\mathbf{v}_j, \dots, \mathbf{v}_j, \mathbf{v}_{j'}, \dots, \mathbf{v}_0) - \beta_\alpha^{(j)}(\mathbf{v}_j, \dots, \mathbf{v}_j, \mathbf{v}_j, \dots, \mathbf{v}_0) \quad (2.49)$$

we can decompose  $r_\alpha^{(j)}(\mathbf{v}_j, \dots, \mathbf{v}_0)$  in the following way:

$$r_{\alpha,j}(\mathbf{v}_j, \dots, \mathbf{v}_0) = \sum_{j'=j+1}^0 D_\alpha^{(j,j')}(\mathbf{v}_j, \dots, \mathbf{v}_0) \quad (2.50)$$

Note that  $D_\alpha^{(j,j')}(\mathbf{v}_j, \dots, \mathbf{v}_0)$  is obtained from  $\beta_\alpha^{(j)}(\mathbf{v}_j, \dots, \mathbf{v}_0)$ , by changing the values of the r.c.c. in the following way: the r.c.c. associated to endpoints of scales lower than  $j'$  are put equal to the corresponding r.c.c. of scale  $j$ ; those of scale greater than  $j'$  are left unchanged; at least one of the r.c.c.  $v_{r,j'}$  is substituted with  $v_{r,j'} - v_{r,j}$ . By using the *short memory property* (see e.g. (4.31) of [35]), we can show that, if  $\varepsilon_j$  is small enough,

$$|D_\alpha^{(j,j')}(\mathbf{v}_j, \dots, \mathbf{v}_0)| \leq b_3 \varepsilon_j \gamma^{-(j'-j)\vartheta} |\mathbf{v}_{j'} - \mathbf{v}_j| \quad (2.51)$$

for some  $b_3 > 0$ . If we insert in the flow equation (2.35) the equations (2.41), (2.43), (2.46), (2.50) and use the bounds (2.42), (2.45), (2.47) and (2.51), we get, if  $\varepsilon_j$  is small enough,

$$|\mathbf{v}_{j-1} - \mathbf{v}_j| \leq (a + b_1 \varepsilon_j) |g_{1,j}|^2 + (\bar{c} \varepsilon_0 + b_2 \varepsilon_j^2) \gamma^{\vartheta j} + b_3 \varepsilon_j \sum_{j'=j+1}^0 \gamma^{-\vartheta(j'-j)} |\mathbf{v}_{j'} - \mathbf{v}_j| \quad (2.52)$$

for some  $b_2 > 0$ . The form of this bound implies that, in order to control the flow, it is sufficient to prove that  $g_{1,j}$  goes to 0 as  $j \rightarrow -\infty$  so fast that  $|g_{1,j}|^2$  is summable on  $j$ . Hence, we have to look more carefully to the flow equation of  $g_{i,j}$ . By proceeding as before, we can write

$$g_{1,j-1} = g_{1,j} - a g_{1,j}^2 + \tilde{r}_{1,j} + r_{1,j} + \bar{r}_{1,j} \quad (2.53)$$

$$|\bar{r}_{1,j}| \leq b_2 \varepsilon_j^2 \gamma^{\vartheta j}, \quad |\tilde{r}_{1,j}| \leq b_1 \varepsilon_j |g_{1,j}|^2, \quad |r_{1,j}| \leq b_3 \varepsilon_j \sum_{j'=j+1}^0 \gamma^{-\vartheta(j'-j)} |\mathbf{v}_{j'} - \mathbf{v}_j| \quad (2.54)$$

It is easy to show that, if  $\varepsilon_0$  is small enough, there is a constant  $c_4$ , such that, if  $g_{1,0} \in D_{\varepsilon_0,\delta}$  and  $c_4 |j_0| |g_{1,0}|^2 \leq |g_{1,0}|^{2-\eta}$ ,  $\eta < 1$ , then, for  $j \geq j_0$ ,

$$g_{1,j} \in D_{2\varepsilon_0,\delta/2}, \quad |g_{1,0}|/2 \leq |g_{1,j}| \leq 2|g_{1,0}|, \quad \varepsilon_j \leq 2\varepsilon_0 \quad (2.55)$$

Hence we put  $j_0 = -(c_4 |g_{1,0}|^{1/2})^{-1}$  and suppose  $\varepsilon_0$  so small that

$$\varepsilon_{j_0} \gamma^{\frac{\vartheta}{2} j_0} \leq 2c_5 |g_{1,j_0}| \gamma^{\frac{\vartheta}{2} j_0} \leq |g_{1,j_0}|^3 \quad (2.56)$$

where we also used the fact that, since  $\hat{v}(2p_F) > 0$ ,  $\varepsilon_0 \leq c_5 |g_{1,0}|$ , for some constant  $c_5$ .

**Lemma 2.4** *If  $g_{1,0} \in D_{\varepsilon_0, \delta}$  and  $j \geq j_0$ , then, if  $\varepsilon_0$  is small enough,*

$$|\mathbf{v}_{j-1} - \mathbf{v}_j| \leq 2a|g_{1,j}|^2 + 2\bar{c}\varepsilon_0\gamma^{\frac{\vartheta}{2}j} \quad (2.57)$$

**Proof** - We shall proceed by induction. By (2.55), if  $\varepsilon_0$  is small enough,  $\bar{c}\varepsilon_0 + b_2\varepsilon_j^2 \leq (3/2)\bar{c}\varepsilon_0$  and  $a + b_1\varepsilon_j \leq 3a/2$ ; hence, (2.57) is true for  $j = 0$ . Let us suppose that (2.57) is verified for  $j > h > 0$ . By (2.55), if  $j \geq h \geq j_0$ ,  $|g_{1,j}|/|g_{1,h}| \leq 4$ ; hence, by using (2.52) and (2.57), we get:

$$\begin{aligned} |\mathbf{v}_{h-1} - \mathbf{v}_h| &\leq (3/2)a|g_{1,h}|^2 + (3/2)\bar{c}\varepsilon_0\gamma^{\vartheta h} + \\ &b_3\varepsilon_h \sum_{j=h+1}^0 \gamma^{-\vartheta(j-h)}(j-h) \max_{h < j' \leq j} \left[ 2a|g_{1,j'}|^2 + 2\bar{c}\varepsilon_0\gamma^{\frac{\vartheta}{2}j'} \right] \\ &\leq |g_{1,h}|^2 \left[ (3/2)a + 64ab_3\varepsilon_0 \sum_{n=0}^{\infty} n\gamma^{-\vartheta n} \right] + \gamma^{\frac{\vartheta}{2}h}\varepsilon_0 \left[ (3/2)\bar{c} + 4\bar{c}b_3\varepsilon_0 \sum_{n=0}^{\infty} n\gamma^{-\frac{\vartheta}{2}n} \right] \end{aligned}$$

Hence, (2.57) is verified also for  $j = h$ , if  $\varepsilon_0$  is small enough. ■

The previous analysis implies that the flow is essentially trivial up to values of  $j$  of order  $|g_{1,0}|^{-1/2}$  (or even  $|g_{1,0}|^{-\eta}$ ,  $0 < \eta < 1$ ). If  $j \leq j_0$ , we write (2.53) in the form

$$g_{1,j-1} = g_{1,j} - a_j g_{1,j}^2, \quad a_j \equiv a - \frac{\tilde{r}_{1,j} + r_{1,j} + \bar{r}_{1,j}}{g_{1,j}^2} \quad (2.58)$$

and we define  $A_{j_0} = 0$  and, for  $j < j_0$ ,

$$A_j = \frac{1}{j_0 - j} \sum_{j'=j+1}^{j_0} a_{j'} \quad \tilde{g}_{1,j} = \frac{g_{1,j_0}}{1 + A_j g_{1,j_0}(j_0 - j)} \quad (2.59)$$

**Lemma 2.5** *There are constants  $c_1, c_2, c_3$  such that, if  $g_{1,0} \in D_{\varepsilon_0, \delta}$  and it  $\varepsilon_0$  is small enough, then the following bounds are satisfied, for all  $j < j_0$ .*

$$\varepsilon_j \leq c_3\varepsilon_0 \quad (2.60)$$

$$|\mathbf{v}_j - \mathbf{v}_{j+1}| \leq c_1|g_{1,j+1}|^2 \quad (2.61)$$

$$|g_{1,j} - \tilde{g}_{1,j}| \leq |\tilde{g}_{1,j}|^{3/2} \quad (2.62)$$

$$|a_j - a| \leq c_2|g_{1,j_0}| \quad (2.63)$$

**Proof** - We shall proceed by induction. By using (2.56), (2.57) and (2.55), we see that the bounds (2.60) and (2.61) are satisfied for  $j = j_0$ , if  $c_3 \geq 2$ ,  $c_1 \geq 3a$  and  $4\bar{c}\varepsilon_0 \leq a$ . Moreover,  $g_{1,j_0} = \tilde{g}_{1,j_0}$  and, by proceeding as in the proof of Lemma 2.4 and using (2.56), it is easy to prove that there is a constant  $\bar{c}_2$ , such that

$$|a_{j_0} - a| \leq \bar{c}_2|g_{1,j_0}|$$

Hence, all the bounds are verified (for  $\varepsilon_0$  small enough) for  $j = j_0$ , if  $c_1 \geq 3a$ ,  $c_2 \geq \bar{c}_2$  and  $c_3 \geq 2$ . Suppose that they are verified for  $j_0 \geq j \geq h$ .

The validity of (2.62) for  $j = h - 1$  follows from Prop. A.2, which only rests on the bound (2.63) for  $j \geq h$ . On the other hand, (2.62) implies that, if  $\varepsilon_0$  is small enough,  $2^{-1}|\tilde{g}_{1,j}| \leq |g_{1,j}| \leq 2|\tilde{g}_{1,j}|$ ; hence, using (2.59), we get, for  $j > h$

$$\left| \frac{g_{1,j}}{g_{1,h}} \right| \leq 4 \frac{|1 + A_h g_{1,j_0}(j_0 - h)|}{|1 + A_j g_{1,j_0}(j_0 - j)|} \quad (2.64)$$

Let us now define, as in App. A,  $A_j = \alpha_j + i\beta_j$ ,  $\alpha_j = \Re A_j$ , and suppose that

$$2c_2\varepsilon_0 \leq a/2 \quad (2.65)$$

so that, by (2.55),  $\alpha_j \geq a/2$ ,  $|\beta_j| \leq 2c_2\varepsilon_0$ ,  $|A_j| \leq 3a/2$ , for  $j > h$ . By proceeding as in the proof of the bound (A.8) in App. A, we get, if  $j > h$  and  $|\arg g_{1,0}| \leq \pi - \delta$ ,  $\delta > 0$  (so that  $|\arg g_{1,j_0}| \leq \pi - \delta/2$ , see (2.55)),

$$|1 + g_{1,j_0}\alpha_j(j_0 - j)| \geq \frac{1}{3} \sin(\delta/2)[1 + |g_{1,j_0}|\alpha_j(j_0 - j)]$$

and, if we put  $1 + A_j g_{1,j_0}(j_0 - j) = 1 + \alpha_j g_{1,j_0}(j_0 - j) + w_j$ , we choose  $\varepsilon_0$  so that

$$\frac{|w_j|}{|1 + g_{1,j_0}\alpha_j(j_0 - j)|} \leq \frac{6c_2\varepsilon_0|g_{1,j_0}|(j_0 - j)}{\sin(\delta/2)|g_{1,j_0}|(a/2)(j_0 - j)} = \frac{12c_2\varepsilon_0}{a \sin(\delta/2)} \leq \frac{1}{2} \quad (2.66)$$

Then, by using (2.64), we get

$$\left| \frac{g_{1,j}}{g_{1,h}} \right| \leq \frac{24}{\sin(\delta/2)} \frac{1 + (3a/2)|g_{1,j_0}|(j_0 - h)}{1 + (a/2)|g_{1,j_0}|(j_0 - j)} \leq C_\delta(j - h) \quad (2.67)$$

for some constant  $C_\delta$ , only depending on  $\delta$  and  $a$ . Moreover, since  $\varepsilon_h \leq c_3\varepsilon_0$ , then  $\bar{c}\varepsilon_0 + b_2\varepsilon_h^2 \leq 2\bar{c}\varepsilon_0$  and  $a + b_1\varepsilon_j \leq 2a$ , if

$$b_2c_3^2\varepsilon_0 \leq \bar{c}, \quad \text{and } b_1c_3\varepsilon_0 \leq a \quad (2.68)$$

Hence, by using the bounds (2.52), (2.61), (2.56) and (2.67), we get

$$\begin{aligned} |\mathbf{v}_{h-1} - \mathbf{v}_h| &\leq 2a|g_{1,h}|^2 + 2\bar{c}\varepsilon_0\gamma^{-\vartheta(j_0-h)}|g_{1,j_0}|^2 + c_1b_3\varepsilon_h \sum_{j=h+1}^0 \gamma^{-\vartheta(j-h)}(j-h) \max_{h < j' \leq j} |g_{1,j'}|^2 \\ &\leq |g_{1,h}|^2 \left[ 2a + 2\bar{c}\varepsilon_0C_\delta^2 \max_{n \geq 0} \gamma^{-n\vartheta} n^2 + c_1\varepsilon_h b_3 C_\delta^2 \sum_{n=0}^{\infty} \gamma^{-\vartheta n} n^3 \right] \end{aligned}$$

It follows that (2.61) is satisfied also for  $j = h$ , if

$$2a + 2\bar{c}\varepsilon_0C_\delta^2 \max_{n \geq 0} \gamma^{-n\vartheta} n^2 + 2c_1c_3\varepsilon_0b_3C_\delta^2 \sum_{n=0}^{\infty} \gamma^{-\vartheta n} n^3 \leq c_1 \quad (2.69)$$

Moreover, by using (2.61) and  $|g_{1,j}| \leq 2|\tilde{g}_{1,j}|$ , we get, for some  $b_4 > 0$ , only depending on  $a$ , under the condition (2.65):

$$\varepsilon_{h-1} \leq \varepsilon_0 + \sum_{j=h}^0 |\mathbf{v}_{j-1} - \mathbf{v}_j| \leq \varepsilon_0 + b_4c_1\varepsilon_0$$

so that  $\varepsilon_{h-1} \leq c_3\varepsilon_0$ , if

$$1 + b_4c_1 \leq c_3 \quad (2.70)$$

The bound for  $a_{h-1} - a$  can be done in the same way; it is easy to see that

$$|a_{h-1} - a| \leq \left[ b_1c_3 + b_2c_3^2\varepsilon_0C_\delta^2 \max_{n \geq 0} \gamma^{-n\vartheta} n^2 + 2c_1c_3b_3C_\delta^2 \sum_{n=0}^{\infty} \gamma^{-\vartheta n} n^3 \right] \varepsilon_0 \quad (2.71)$$

Hence, (2.63) is verified for  $j = h - 1$ , if

$$\tilde{c}_2 \equiv 2\bar{c}C_\delta^2 \max_{n \geq 0} \gamma^{-n\vartheta} n^2 + 2c_1c_3b_3C_\delta^2 \sum_{n=0}^{\infty} \gamma^{-\vartheta n} n^3 \leq c_2 \quad (2.72)$$

The conditions (2.65), (2.66), (2.68), (2.69), (2.70) and (2.72) can be all satisfied, by taking, for example,  $c_1 = 4a$ ,  $c_3 = 1 + 4ab_4$  and  $c_2 = \max\{\bar{c}_2, \tilde{c}_2\}$ , if  $\varepsilon_0$  is small enough.  $\blacksquare$

Lemma 2.5 implies that, if  $g_{1,0} \in D_{\varepsilon,\delta}$ , with  $\varepsilon$  small enough (how small depending on  $\delta$ ),  $g_{1,j}$  goes to 0, as  $j \rightarrow -\infty$ , and  $\sum_{j=h}^0 |g_{1,j}|^2 \leq C\delta^{-1}|\lambda|$ , uniformly in  $h$ . This is an easy consequence of the condition (2.62) and the condition  $\hat{v}(2p_F) > 0$ ; note that the power  $3/2$  in the r.h.s. of (2.62) could be replaced  $2 - \eta$ ,  $\eta > 0$ , but 2 is not allowed. The form of the flow (2.35) then implies also that  $g_{2,j}$ ,  $g_{4,j}$  and  $\delta_j$  converge, as  $j \rightarrow -\infty$ , to some limits  $g_{2,-\infty}$ ,  $g_{4,-\infty}$  and  $\delta_{-\infty}$  of order  $\lambda$ , such that

$$g_{2,-\infty} = g_{2,0} - \frac{1}{2}g_{1,0} + O(|\lambda|^{3/2}) = [2\hat{v}(0) - \hat{v}(2p_F)]\lambda + O(|\lambda|^{3/2}) \quad (2.73)$$

$$\begin{aligned} g_{4,-\infty} &= g_{4,0} + O(\lambda^2) = 2\lambda\hat{v}(0) + O(\lambda^2) \\ \delta_{-\infty} &= O(\lambda) \end{aligned} \quad (2.74)$$

Let us now suppose that  $\lambda$  is a (small) positive number; the previous bounds imply that  $g_{1,j} > 0$ , for any  $j \leq 0$ . The following Lemma will allow to control the logarithmic corrections to the power law fall-off of the correlations.

**Lemma 2.6** *There are four sequences  $w_{i,h}$ ,  $\delta_{i,h}$ ,  $i = 1, 2$ ,  $h \leq j_0$ , such that*

$$\sum_{j=h}^{j_0} g_{1,j} = (1 + w_{1,h})\frac{1}{a} \log[1 + ag_{1,j_0}(j_0 - h)] + \delta_{1,h} \quad (2.75)$$

$$\sum_{j=h}^{j_0} [g_{2,j} - g_{2,-\infty}] = (1 + w_{2,h})\frac{1}{2a} \log[1 + ag_{1,j_0}(j_0 - h)] + \delta_{2,h} \quad (2.76)$$

with

$$|w_{i,h}| \leq C\lambda, \quad |\delta_{i,h}| \leq C\lambda^{1/2} \quad (2.77)$$

$$|w_{i,h-1} - w_{i,h}| \leq \frac{C\lambda}{[1 + ag_{1,j_0}(j_0 - h)] \log[1 + ag_{1,j_0}(j_0 - h)]} \quad (2.78)$$

**Proof** - Let us put  $g_0 = g_{1,j_0}$ , and  $a(s)$  the function of  $s \geq 0$ , such that  $a(s) = a_{j_0-n}$ , if  $n \leq s < n+1$ . Then, by using (2.59), (2.62) and (2.63), it is easy to see that

$$\left| \sum_{j=h}^{j_0} g_{1,j} - I_{j_0-h} \right| \leq C\lambda^{1/2}, \quad I_n = \int_0^n ds \frac{g_0}{1 + g_0 \int_0^s dt a(t)} \quad (2.79)$$

On the other hand, (2.63) also implies that  $a(s) = a + \lambda r(s)$ , with  $|r(s)| \leq C$ ; hence

$$I_n = \int_0^n ds \frac{g_0}{1 + g_0 a s} - \lambda \int_0^n ds \frac{g_0^2 \int_0^s dt r(t)}{[1 + g_0 \int_0^s dt a(t)][1 + g_0 a s]}$$

implying that

$$\left| I_n - \frac{1}{a} \log(1 + ag_0 n) \right| \leq \frac{4C\lambda}{a^2} \int_0^{ag_0 n} dx \frac{x}{(1+x)^2} < \frac{4C\lambda}{a^2} \log(1 + ag_0 n)$$

Hence there is a constant  $\tilde{w}_n$  such that  $I_n = (1/a + \tilde{w}_n) \log(1 + ag_0 n)$ , with  $|\tilde{w}_n| \leq C\lambda$ ; this bound, together with the bound in (2.79), proves (2.77) for  $i = 1$ . To prove (2.78), note that

$$|I_{n+1} - I_n| \leq \int_n^{n+1} ds \frac{g_0}{1 + g_0 \frac{a}{2}s} = \frac{2}{a} \log \left( 1 + \frac{ag_0}{2 + ag_0 n} \right)$$

$$I_{n+1} - I_n = (1/a + \tilde{w}_{n+1}) \log \left( 1 + \frac{ag_0}{1 + ag_0 n} \right) + (\tilde{w}_{n+1} - \tilde{w}_n) \log(1 + ag_0 n)$$

so that, if  $\lambda$  is small enough,

$$|\tilde{w}_{n+1} - \tilde{w}_n| \log(1 + ag_0 n) \leq \left( \frac{3}{a} + C\lambda \right) \log \left( 1 + \frac{ag_0}{1 + ag_0 n} \right) \leq \frac{4g_0}{1 + ag_0 n}$$

To prove (2.77) and (2.78) for  $i = 2$ , note that, by (2.46) and Lemma 2.5, if  $j \leq j_0$ ,

$$\begin{aligned} g_{2,j} - g_{2,-\infty} &= \sum_{h=-\infty}^j \left[ \frac{a}{2} + O(\lambda) \right] g_{1,h}^2 = \left[ \frac{a}{2} + O(\lambda) \right] \int_{|j|}^{\infty} ds \frac{g_0^2}{(1 + ag_0 s)^2} + O(\tilde{g}_{1,j}^{3/2}) \\ &= \left[ \frac{1}{2} + O(\lambda) \right] \tilde{g}_{1,j} + O(\tilde{g}_{1,j}^{3/2}) \end{aligned}$$

Hence, the proof of (2.77) is almost equal to the previous one, while the proof of (2.78) needs a slightly different algebra; we omit the details. ■

## 2.4 The flow of renormalization constants

The renormalization constant of the free measure satisfies

$$\frac{Z_{j-1}}{Z_j} = 1 + \beta_z^{(j)}(\vec{g}_j, \delta_j, \dots, \vec{g}_0, \delta_0) + \bar{\beta}_z^{(j)}(\vec{v}_j; \dots, \vec{v}_0; \lambda); \quad (2.80)$$

while the renormalization constants of the densities, for  $\alpha = C, S_i, SC, TC_i$  and  $i = 1, 2$ , satisfy the equations

$$\frac{Z_{j-1}^{(i,\alpha)}}{Z_j^{(i,\alpha)}} = 1 + \beta_{(i,\alpha)}^{(j)}(\vec{g}_j, \delta_j, \dots, \vec{g}_0, \delta_0) + \bar{\beta}_{(i,\alpha)}^{(j)}(\vec{v}_j; \dots, \vec{v}_0; \lambda). \quad (2.81)$$

In these two formulas, by definition, the  $\beta_t^{(j)}$  functions, with  $t = z$  or  $(i, \alpha)$ , are given by a sum of multiscale graphs, containing only vertices with r.c.c.  $\vec{g}_h, \delta_h$ ,  $0 \geq h \geq j$ , modified so that the propagators  $g_\omega^{(h)}$  and the renormalization constants  $Z_h$ ,  $Z_h^{(i,\alpha)}$ ,  $0 \geq h \geq j$ , are replaced by  $g_{D,\omega}^{(h)}$ ,  $Z_h^{(D)}$ ,  $Z_h^{(D,i,\alpha)}$  (the definition of  $Z_h^{(D,i,\alpha)}$  is analogue to the one of  $Z_h^{(D)}$ ); the  $\bar{\beta}_t^{(j)}$  functions contain the correction terms together the remainder of the expansion. Note that, by definition, the constants  $Z_j^{(D)}$  are exactly those generated by (2.80) and (2.41) with  $\bar{\beta}_z^{(j)} = \bar{\beta}_\alpha^{(j)} = 0$ . Note also that  $|\bar{\beta}_z^{(j)}| \leq C\bar{v}_j^2 \gamma^{\vartheta j}$ , while  $|\bar{\beta}_{(i,\alpha)}^{(j)}| \leq C\bar{v}_j \gamma^{\vartheta j}$ .

By using (2.80) and (2.81), we can write

$$\frac{Z_{j-1}^{(1,\alpha)}}{Z_j} = \frac{Z_j^{(1,\alpha)}}{Z_j} \left[ 1 + \beta_{z,(1,\alpha)}^{(j)}(\vec{g}_j, \delta_j) + \hat{\beta}_{z,(1,\alpha)}^{(j)}(\vec{v}_j; \dots, \vec{v}_0; \lambda) \right] \quad (2.82)$$

with  $|\hat{\beta}_{z,(1,\alpha)}| \leq C\bar{v}_j \gamma^{\vartheta j}$ . If we define  $\tilde{\beta}_{z,(1,\alpha)}^{(j)}(\vec{g}, \delta)$  the value of  $\beta_{z,(1,\alpha)}^{(j)}(\vec{g}_j, \delta_j; \dots, \vec{g}_0, \delta_0)$  at  $(\vec{g}_i, \delta_i) = (\vec{g}, \delta)$ ,  $j \leq i \leq 0$  and  $\tilde{\beta}_{z,(1,\alpha)}^{(j,\leq 1)}(\vec{g}, \delta)$  the sum of its terms of order 0 and 1 in  $g_{1,h}$ , it turns out that

$$|\tilde{\beta}_{1,\alpha}^{(1)(j)}(\vec{g}_j, \delta_j)| \leq C[\max\{|g_1|, |g_2|, |g_4|, |\delta|\}]^2 \gamma^{\vartheta h}, \quad \text{if } \alpha = C \quad (2.83)$$

This bound, as crucial as the analogous bound (2.45), has been proved in [33]; the proof will be sketched in App. C. The bound (2.83), together with  $\sum_{k=j}^0 |g_{1,k}|^2 \leq C|\lambda|$  and the fact that  $Z_h^{(1,S_i)} = Z_h^{(1,C)}$  by the SU(2) spin symmetry, imply that

$$\left| \frac{Z_j^{(1,\alpha)}}{Z_j} - 1 \right| \leq C|\bar{\varepsilon}_j^2|, \quad \alpha = C, S_i \quad (2.84)$$

Regarding the flow of the other renormalization constants, we can write

$$Z_j^{(t)} = \gamma^{-\eta_{t,j}} \hat{Z}_j^{(t)} \quad (2.85)$$

where  $Z_j^{(z)} = Z_j$  and, by definition,

$$\eta_t \equiv \lim_{j \rightarrow -\infty} \eta_{t,j} := \log_\gamma \left[ 1 + \beta_t^{(0,j)}(g_{2,-\infty}, g_{4,-\infty}, \delta_{-\infty}; \dots; g_{2,-\infty}, g_{4,-\infty}, \delta_{-\infty}) \right] \quad (2.86)$$

Note that the exponents  $\eta_t$  are functions of  $\vec{v}_{-\infty}$  only, an observation which will play a crucial role in the following. Moreover, by an explicit first order calculation, we see that

$$\eta_t = \begin{cases} (2\pi v_F)^{-1} g_{2,-\infty} + O(\lambda^2) & t = (2, C), (2, S_i) \\ -(2\pi v_F)^{-1} g_{2,-\infty} + O(\lambda^2) & t = (2, SC), (2, TC_i) \\ O(\lambda^2) & \text{otherwise} \end{cases} \quad (2.87)$$

while

$$\begin{aligned} \frac{\hat{Z}_{h-1}^{(t)}}{\hat{Z}_h^{(t)}} &= 1 + O(\tilde{g}_{1,h}\lambda) + r_h^{(t)}, \quad t = z, (1, \alpha), \alpha \neq TC_i \\ \frac{\hat{Z}_{h-1}^{(2,C)}}{\hat{Z}_h^{(2,C)}} &= 1 - ag_{1,h} + \frac{a}{2}(g_{2,h} - g_{2,-\infty}) + O(\tilde{g}_{1,h}\lambda) + r_h^{(2,C)} \\ \frac{\hat{Z}_{h-1}^{(2,S_i)}}{\hat{Z}_h^{(2,S_i)}} &= 1 + \frac{a}{2}(g_{2,h} - g_{2,-\infty}) + O(\tilde{g}_{1,h}\lambda) + r_h^{(2,S_i)} \\ \frac{\hat{Z}_{h-1}^{(2,SC)}}{\hat{Z}_h^{(2,SC)}} &= 1 - \frac{a}{2}g_{1,h} - \frac{a}{2}(g_{2,h} - g_{2,-\infty}) + O(\tilde{g}_{1,h}\lambda) + r_h^{(2,SC)} \\ \frac{\hat{Z}_{h-1}^{(2,TC_i)}}{\hat{Z}_h^{(2,TC_i)}} &= 1 + \frac{a}{2}g_{1,h} - \frac{a}{2}(g_{2,h} - g_{2,-\infty}) + O(\tilde{g}_{1,h}\lambda) + r_h^{(2,TC_i)} \end{aligned} \quad (2.88)$$

where  $a$  and  $\tilde{g}_{1,h}$  are defined as in (2.48) and (2.59), respectively, and  $\sum_{h=-\infty}^0 |r_h^{(t)}| \leq C|\lambda|^2$ . Let us define:

$$q_t^{(h)} = \frac{\log \hat{Z}_h^{(t)}}{\log(1 + ag_{1,0}|h|)} \quad (2.89)$$

Hence, by using (2.75), (2.76) and (2.77), we get

$$\begin{aligned} |q_t^{(h)}| &\leq C\lambda, \quad t = z, (1, \alpha), \alpha \neq TC_i \\ |q_t^{(h)} - \frac{1}{2}\bar{\zeta}_\alpha| &\leq C\lambda, \quad t = (2, \alpha) \end{aligned} \quad (2.90)$$

where the constants  $\bar{\zeta}_\alpha$  are those of (1.20).

## 2.5 Proof of Theorem 1.1

In order to prove the representation (1.19) of the density correlations  $\Omega_\alpha(\mathbf{x})$ , we can proceed exactly as in §5 of [35], where a similar problem was treated in all details; hence we shall only describe the result. We can write for  $\Omega_\alpha(\mathbf{x} - \mathbf{y})$  a convergent tree expansion, whose trees have two endpoints associated to density operators (*special endpoints*) and an arbitrary number of interaction endpoints (*normal endpoints*). As one could expect,  $\Omega_\alpha(\mathbf{x} - \mathbf{y})$  behaves, as  $|\mathbf{x} - \mathbf{y}| \rightarrow \infty$ , as the function  $\tilde{\Omega}_\alpha(\mathbf{x} - \mathbf{y})$  calculated by taking only the trees with two special endpoints of scale  $h \leq 0$  and no normal endpoints. This function can be obtained by the following procedure. Let us consider the expression

$$\langle O_{\mathbf{x}}^{(1,\alpha)}; O_{\mathbf{y}}^{(1,\alpha)} \rangle_{\mathbf{D}}^T + \langle O_{\mathbf{x}}^{(2,\alpha)}; O_{\mathbf{y}}^{(2,\alpha)} \rangle_{\mathbf{D}}^T$$

where the operators  $O_{\mathbf{x}}^{(i,\alpha)}$ ,  $i = 1, 2$ , are those defined in (2.28) and (2.29) and  $\langle \cdot \rangle_{\mathbf{D}}^T$  is the truncated expectation evaluated with covariance  $\sum_h Z_h^{-1} g_{\mathbf{D},\omega}^{(h)}$ . Hence, this expression can be written as a sum of terms, each one proportional to  $Z_h^{-1} g_{\mathbf{D},\omega}^{(h)}(\mathbf{x} - \mathbf{y}) Z_{h'}^{-1} g_{\mathbf{D},\omega'}^{(h')}(\mathbf{x} - \mathbf{y})$ , for some values of  $h$ ,  $h'$ ,  $\omega$  and  $\omega'$ .  $\tilde{\Omega}_\alpha(\mathbf{x} - \mathbf{y})$  is obtained by multiplying each one of these terms by  $[Z_{h \vee h'}^{(1,\alpha)}]^2$  ( $h \vee h' = \max\{h, h'\}$ ), if it appears in the calculation of  $\langle O_{\mathbf{x}}^{(1,\alpha)} O_{\mathbf{y}}^{(1,\alpha)} \rangle_{\mathbf{D}}^T$ , otherwise by  $[Z_{h \vee h'}^{(2,\alpha)}]^2$ . Let us consider first the case  $\alpha = C$ ; we have

$$\begin{aligned} \tilde{\Omega}_C(\mathbf{x}) &= \Omega^{(1,C)}(\mathbf{x}) + \cos(2p_F x) \Omega^{(2,C)}(\mathbf{x}) \\ \Omega^{(1,C)}(\mathbf{x}) &= 2 \sum_{\omega} \sum_{h,h'} \frac{[Z_{h \vee h'}^{(1,C)}]^2}{Z_h Z_{h'}} g_{\mathbf{D},\omega}^{(h)}(\mathbf{x}) g_{\mathbf{D},\omega}^{(h')}(\mathbf{x}) \\ \Omega^{(2,C)}(\mathbf{x}) &= 4 \sum_{h,h'} \frac{[Z_{h \vee h'}^{(2,C)}]^2}{Z_h Z_{h'}} g_{\mathbf{D},+}^{(h)}(\mathbf{x}) g_{\mathbf{D},-}^{(h')}(\mathbf{x}) \end{aligned} \quad (2.91)$$

Let us now observe that, for any  $N > 0$ ,  $|g_{\mathbf{D},\omega}^{(h)}(\mathbf{x})| \leq C_N \gamma^h [1 + (\gamma^h |\mathbf{x}|)^N]^{-1}$ . Hence, if  $|\mathbf{x}| \geq 1$ , in the previous sums the main contribution is given by the terms with  $|h|$  and  $|h'|$  of the same size as  $\log_\gamma |\mathbf{x}|$ , so that one expects that the asymptotic behavior of  $\Omega^{(i,C)}(\mathbf{x})$ ,  $i = 1, 2$ , is the same of the the function  $\bar{\Omega}^{(i,C)}(\mathbf{x})$ , obtained by the substitutions of  $\gamma^{-h}$  and  $\gamma^{-h'}$  with  $|\mathbf{x}|$  in the asymptotic expressions of the renormalization constants, given by (2.85) and (2.88), that is

$$\frac{[Z_{h \vee h'}^{(i,C)}]^2}{Z_h Z_{h'}} \rightarrow |\mathbf{x}|^{2(\eta_{i,C} - \eta_z)} \left[ 1 + f(\lambda) \log |\mathbf{x}| \right]^{2(q_{i,C}^{(h_{\mathbf{x}})} - q_z^{(h_{\mathbf{x}})})} \quad (2.92)$$

where the coefficients  $q_t^{(h)}$  are defined as in (2.89),  $h_{\mathbf{x}} = \inf\{h : \gamma^h |\mathbf{x}| \geq 1\}$ , and, by (2.26), (2.48), (2.59) and Lemma 2.6,

$$f(\lambda) = \frac{ag_{1,j_0}}{\log \gamma} = \frac{2\lambda \hat{v}(2p_F)}{\pi v_F} + O(\lambda^{3/2}) \quad (2.93)$$

In order to justify the substitution (2.92), let us put  $\eta_i = 2(\eta_{i,C} - \eta_z)$  and  $q_i(\mathbf{x})$  any continuous interpolation between  $2[q_{i,C}^{(h_{\mathbf{x}})} - q_z^{(h_{\mathbf{x}})}]$  and  $2[q_{i,C}^{(h_{\mathbf{x}}-1)} - q_z^{(h_{\mathbf{x}}-1)}]$ . Note that, thanks to the bounds (2.77) and (2.78),  $q_i(\mathbf{x})$  is a bounded function of order  $\lambda$ , defined up to fluctuations bounded, for  $|\mathbf{x}| \geq 1$ , by  $C\lambda[L(\mathbf{x}) \log L(\mathbf{x})]^{-1}$ , with  $L(\mathbf{x}) = 1 + f(\lambda) \log |\mathbf{x}|$ ; hence, its precise definition



modifies the following expressions only for a factor  $1 + O(\lambda)$ . Let us now note that

$$|\Omega^{(i,C)}(\mathbf{x}) - \bar{\Omega}^{(i,C)}(\mathbf{x})| \leq C_N |\mathbf{x}|^{\eta_i-2} [1 + f(\lambda) \log |\mathbf{x}|]^{q_i(\mathbf{x})} \sum_{h,h'} \frac{\gamma^h |\mathbf{x}|}{1 + (\gamma^h |\mathbf{x}|)^N} \frac{\gamma^{h'} |\mathbf{x}|}{1 + (\gamma^{h'} |\mathbf{x}|)^N} \cdot$$

$$\cdot \left| \frac{(\gamma^h |\mathbf{x}|)^{\eta_z} (\gamma^{h'} |\mathbf{x}|)^{\eta_z}}{(\gamma^{h \vee h'} |\mathbf{x}|)^{2\eta_{i,C}}} \left[ \frac{L(|\mathbf{x}|)}{L(\gamma^{|h|})} \right]^{q_z^{(h)}} \left[ \frac{L(|\mathbf{x}|)}{L(\gamma^{|h'|})} \right]^{q_z^{(h')}} \left[ \frac{L(|\mathbf{x}|)}{L(\gamma^{|h \vee h'|})} \right]^{-2q_{i,C}^{(h \vee h')}} \frac{c_h c_{h'}}{\tilde{c}_{h \vee h'}^2} - 1 \right| \quad (2.94)$$

where

$$L(t) = 1 + f(\lambda) \log t, \quad c_h = L(\gamma^{|h|})^{q_z^{(h)}} / \hat{Z}_h^{(z)}, \quad \tilde{c}_h = L(\gamma^{|h|})^{-q_{i,C}^{(h)}} / \hat{Z}_h^{(i,C)}$$

By (2.88), (2.75) and (2.76),  $c_h = 1 + O(\lambda^{1/2})$  and  $\tilde{c}_h = 1 + O(\lambda^{1/2})$ . On the other hand, if  $r > 0$  and  $t \neq 0$ ,

$$|r^t - 1| \leq |t \log r| (r^t + r^{-t})$$

and, if  $q \neq 0$ ,

$$\left| \left[ \frac{L(|\mathbf{x}|)}{L(\gamma^{|h|})} \right]^q - 1 \right| \leq C_q \left[ |f(\lambda) \log(\gamma^h |\mathbf{x}|)| + |f(\lambda) \log(\gamma^h |\mathbf{x}|)|^{|q|+1} \right]$$

These two bounds, together with the bound

$$\sum_{h=-\infty}^0 \frac{(\gamma^h r)^\alpha |\log(\gamma^h r)|^\beta}{1 + (\gamma^h r)^N} \leq C_{N,\alpha,q}$$

valid for any  $\beta, r > 0, a > 0$  and  $N > \alpha$ , imply that

$$|\Omega^{(i,C)}(\mathbf{x}) - \bar{\Omega}^{(i,C)}(\mathbf{x})| \leq C_N \lambda^{1/2} |\mathbf{x}|^{\eta_i-2} [1 + f(\lambda) \log |\mathbf{x}|]^{q_i(\mathbf{x})} \quad (2.95)$$

By the remark after (2.59), the factor  $\lambda^{1/2}$  can be improved up to  $\lambda^{1-\vartheta}$ ,  $\vartheta < 1$ .

By proceeding as in §5 of [35], it is possible to prove that a bound of this type is satisfied also from the sum over all the other trees. Hence, in order to complete the proof of (1.19) in the case  $\alpha = C$ , we have only to calculate  $\bar{\Omega}^{(1,C)}(\mathbf{x})$  and  $\bar{\Omega}^{(2,C)}(\mathbf{x})$ . By using (2.84), we see that  $\eta_{1,C} = \eta_z$  and  $q_{1,C}^{(h)} = q_z^{(h)}$ , so that, if we define  $X_C = 1 - \eta_{2,C} - \eta_z$  and  $\zeta_C(\mathbf{x}) = 2[q_{2,C}(\mathbf{x}) - q_z(\mathbf{x})]$ , we get

$$\bar{\Omega}^{(1,C)}(\mathbf{x}) = 2 \sum_{\omega} g_{D,\omega}(\mathbf{x}) g_{D,\omega}(\mathbf{x}) \quad (2.96)$$

$$\bar{\Omega}^{(2,C)}(\mathbf{x}) = 4 |\mathbf{x}|^{2(1-X_C)} [1 + f(\lambda) \log |\mathbf{x}|]^{\zeta_C(\mathbf{x})} g_{D,+}(\mathbf{x}) g_{D,-}(\mathbf{x})$$

where  $g_{D,\omega}(\mathbf{x}) = \sum_{h=-\infty}^0 g_{D,\omega}^{(h)}(\mathbf{x})$ . On the other hand, it is easy to see that, for any  $N \geq 2$ ,

$$g_{D,\omega}(\mathbf{x}) = \frac{1}{2\pi} \frac{1}{v_F x_0 + i\omega x} + O(|\mathbf{x}|^{-N})$$

It follows that, up to terms that we put in the “remainder”  $\hat{R}_C(\mathbf{x})$ ,

$$\bar{\Omega}^{(1,C)}(\mathbf{x}) = \frac{1}{\pi^2 \tilde{\mathbf{x}}^2} \bar{\Omega}_0(\mathbf{x}), \quad \bar{\Omega}^{(2,C)}(\mathbf{x}) = \frac{L(\mathbf{x})^{\zeta_C(\mathbf{x})}}{\pi^2 |\tilde{\mathbf{x}}|^{2X_C}} \quad (2.97)$$

where the functions  $\bar{\Omega}_0(\mathbf{x})$  and  $L(\mathbf{x})$  are defined as in Theorem 1.1. Hence, by using (2.87) and (2.93), we get (1.19) for  $\alpha = C$ , together with the fact that  $\zeta_C(\mathbf{x}) = -3/2 + O(\lambda)$ , in agreement with (1.20), and  $2X_C = 2 - b\lambda + O(\lambda^2)$ , in agreement with (1.22). Note also that, in Theorem

1.1, we have modified the function  $f(\lambda)$  by erasing the terms of order greater than 1 in  $\lambda$ ; the only effect of this modification is a change of the function  $R_C(\mathbf{x})$ , which does not change its bound.

The proof of (1.19) in the other cases is done in the same way. In particular, in the case  $\alpha = S_i$  we have to use again the bound (2.84), while the fact that there is no oscillating contribution to the leading term of  $\Omega_{TC_i}$  is due to the fact there is no local marginal term which can produce it, by the remark after §2.29.

Finally, the proof of the scaling relations (1.22) follows from the important fact, proved in §3.6, that they are the same as those of the effective model. Hence they follow from the explicit calculations of §3.5.

## 2.6 Proof of Theorem 1.4

First consider the free energy (1.4). We can decompose it as  $E(\lambda) = \sum_{h=-\infty}^0 E_h(\lambda)$ , where  $E_h(\lambda)$  is the contribution of the trees whose root has scale  $h$ , hence depends only on the running couplings  $\vec{v}_j$  with scale  $j > h$ . The tree expansion implies that there exists  $\varepsilon_0$ , such that, if

$$\bar{\lambda}_h = \max_{j \geq h} |\vec{v}_j| \leq \varepsilon_0 \quad (2.98)$$

then  $|E_h| \leq c_2 \gamma^{2h} \varepsilon_0$ , with  $c_2$  independent of  $h$ . The analysis of §2.3 implies that, given  $\delta \in (0, \pi/2)$ , there exists  $\varepsilon$  such that, if  $\lambda \in D_{\varepsilon, \delta}$ , the condition (2.98) is verified uniformly in  $h$ ; then it is easy to see that  $E(\lambda)$  is analytic in  $D_{\varepsilon, \delta}$  and continuous in its closure. The domain of analyticity of  $E_h(\lambda)$  is in fact larger; the form of the beta function immediately implies that there exist two constants  $c_3$  and  $\bar{c}$  such that  $\bar{\lambda}_0 \leq c_3 |\lambda|$  and, if  $\bar{\lambda}_h \leq \varepsilon_0$ , then  $\bar{\lambda}_{h-1} \leq \bar{\lambda}_h + \bar{c} \bar{\lambda}_h^2$ ; hence, if  $c_3 |\lambda| \leq \min\{\varepsilon_0/2, 1/[4\bar{c}(|h|+1)]\}$ , then, if  $j > h$  and  $\bar{\lambda}_j \leq 2\bar{\lambda}_0$ ,

$$\bar{\lambda}_{j-1} \leq \bar{\lambda}_0 + |j| \bar{c} \bar{\lambda}_j^2 \leq \bar{\lambda}_0 (1 + 4|j| \bar{c} \bar{\lambda}_0) \leq 2\bar{\lambda}_0$$

It follows that  $E_h(\lambda)$  is analytic in the set (1.31), with  $c_0 = c_3^{-1} \min\{\varepsilon_0/2, 1/(4\bar{c})\}$ , and that  $|E_h(\lambda)| \leq c_1 \gamma^{2h}$ , with  $c_1 = c_2 \varepsilon_0$ ; hence  $E(\lambda)$  satisfies (1.32) with  $\kappa = 2 \log \gamma$ .

Let us now consider the 2-point Schwinger function  $S_2(\mathbf{x})$ . By using the tree expansion (similar to that written in [20] for the infrared part of the spinless continuous Fermi gas), we can write  $S_2(\mathbf{x}) = \sum_{h=-\infty}^0 S_{2,h}(\mathbf{x})$ , where  $S_{2,h}(\mathbf{x})$  is the contribution of the trees whose root has scale  $h$ . By proceeding as in §6 of [20], we can prove that, if (2.98) is verified (possibly with a smaller  $\varepsilon_0$ ), then, for any  $N > 0$ ,

$$|S_{2,h}(\mathbf{x})| \leq c_N \sum_{\bar{h}=h+1}^0 \gamma^{-\frac{\bar{h}-h}{2}} \frac{\gamma^{\bar{h}}}{Z_{\bar{h}}} \frac{1}{1 + (\gamma^{\bar{h}} |\mathbf{x}|)^N}, \quad \left| \frac{1}{Z_h} \right| \leq \gamma^{\frac{|h|}{4}} \quad (2.99)$$

with  $c_N$  independent of  $h$ . Hence, if we define  $h_{\mathbf{x}}$  so that  $\gamma^{h_{\mathbf{x}}} |\mathbf{x}| \in [1, \gamma)$ , then, if  $h_{\mathbf{x}} > h$

$$|S_{2,h}(\mathbf{x})| \leq c_2 \left[ \sum_{\bar{h}=h+1}^{h_{\mathbf{x}}} \gamma^{-\frac{\bar{h}}{2}} \gamma^{\frac{3}{4}\bar{h}} \gamma^{\frac{h}{2}} + \sum_{\bar{h}=h_{\mathbf{x}}}^0 \gamma^{-\frac{\bar{h}}{2}} \gamma^{\frac{3}{4}\bar{h}} \gamma^{\frac{h}{2}} \gamma^{2(h_{\mathbf{x}}-\bar{h})} \right] \leq \tilde{c}_2 \gamma^{\frac{h}{2}} \gamma^{\frac{h_{\mathbf{x}}}{4}} \quad (2.100)$$

and a similar bound holds for  $h_{\mathbf{x}} < h$  so that

$$|S_{2,h}(\mathbf{x})| \leq c_s \gamma^{\frac{h}{2}} (1 + |\mathbf{x}|)^{-1/4} \quad (2.101)$$

and we can proceed as in free energy case, so proving (1.32) for  $S_2(\mathbf{x})$ , with  $c_1 = c_s (1 + |\mathbf{x}|)^{-1/4}$  and  $\frac{\kappa}{\log \gamma} = \frac{1}{2}$  (this value could be improved up to any value smaller than 1, at the price of lowering  $\varepsilon_0$  down to 0).

The previous argument can be extended to the generic  $2n$ -point Schwinger function, by using the same strategy used in §2.3 of [38] to analyze the corresponding tree expansion in the case of the Thirring model. The proof of the Theorem in the case of the density correlations is very similar to that of the 2-point function case, if one uses the description of the tree expansion given in §5 of [35]. We shall not give any further detail; the idea at the base of the proof is always that, in the tree expansion of the correlations at fixed space-time points, the contribution of the trees with the root at scale  $h$  must decrease exponentially with the distance from some fixed scale depending on the space-time points.

### 3 RG Analysis of the Effective Model

#### 3.1 Introduction

We introduce an *effective model* (related to the g-ology model introduced in [12] but differing because the interaction is time-dependent), describing fermions with linear dispersion relation and a non-local interaction; such model will be used to prove the crucial bounds (2.45) and (2.83) (on which the previous analysis is based) and for the proof of the Luttinger liquid relations (1.22), together with Theorems 1.2 and 1.3. The model is expressed in terms of the following Grassmann integral:

$$e^{\mathcal{W}_{[l,N]}(\phi,J)} = \int P_Z(d\psi^{[l,N]}) \exp \left\{ -\tilde{V}(\sqrt{Z}\psi^{[l,N]}) + \sum_{\omega,s} \int d\mathbf{x} J_{\mathbf{x},\omega,s} \psi_{\mathbf{x},\omega,s}^{[l,N]+} \psi_{\mathbf{x},\omega,s}^{[l,N]-} \right. \\ \left. + \sum_{\omega,s} \int d\mathbf{x} [\psi_{\mathbf{x},\omega,s}^{[l,N]+} \phi_{\mathbf{x},\omega,s}^- + \phi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,s}^{[l,N]-}] \right\}, \quad (3.1)$$

where  $\mathbf{x} \in \tilde{\Lambda}$  and  $\tilde{\Lambda}$  is a square subset of  $\mathbb{R}^2$  of size  $\gamma^{-l}$ , say  $\gamma^{-l}/2 \leq |\tilde{\Lambda}| \leq \gamma^{-l}$ ,  $P_Z(d\psi^{[l,N]})$  is the fermionic measure with propagator

$$g_{D,\omega}^{[l,N]}(\mathbf{x} - \mathbf{y}) = \frac{1}{Z} \frac{1}{L^2} \sum_{\mathbf{k}} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{\chi_{[l,N]}(|\tilde{\mathbf{k}}|)}{-ik_0 + \omega ck}, \quad \tilde{\mathbf{k}} = (ck, k_0), \quad c = v_F(1 + \delta) \quad (3.2)$$

where  $Z > 0$  and  $\delta$  are two parameters, to be fixed later,  $v_F$  is defined as in Theorem 1.1 and  $\chi_{l,N}(t)$  is a cut-off function depending on a small positive parameter  $\varepsilon$ , nonvanishing for all  $\mathbf{k}$  and reducing, as  $\varepsilon \rightarrow 0$ , to a compact support function equal to 1 for  $\gamma^l \leq |\mathbf{k}| \leq \gamma^{N+1}$  and vanishing for  $|\mathbf{k}| \leq \gamma^{l-1}$  or  $|\mathbf{k}| \geq \gamma^{N+1}$  (its precise definition can be found in (21) [22]);  $\gamma^l$  is the *infrared cut-off* and  $\gamma^N$  is the *ultraviolet cut-off*. The limit  $N \rightarrow \infty$ , followed from the limit  $l \rightarrow -\infty$ , will be called the *limit of removed cut-offs*. The interaction is

$$\tilde{V}(\psi) = g_{1,\perp} V_{1,\perp}(\psi) + g_{\parallel} V_{\parallel}(\psi) + g_{\perp} V_{\perp}(\psi) + g_4 V_4(\psi) \quad (3.3)$$

with

$$V_{1,\perp}(\psi) = \frac{1}{2} \sum_{\omega,s} \int d\mathbf{x} d\mathbf{y} h(\mathbf{x} - \mathbf{y}) \psi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,-s}^- \psi_{\mathbf{y},-\omega,s}^- \psi_{\mathbf{y},-\omega,-s}^+ \\ V_{\parallel}(\psi) = \frac{1}{2} \sum_{\omega,s} \int d\mathbf{x} d\mathbf{y} h(\mathbf{x} - \mathbf{y}) \psi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,s}^- \psi_{\mathbf{y},-\omega,s}^+ \psi_{\mathbf{y},-\omega,s}^- \\ V_{\perp}(\psi) = \frac{1}{2} \sum_{\omega,s} \int d\mathbf{x} d\mathbf{y} h(\mathbf{x} - \mathbf{y}) \psi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,s}^- \psi_{\mathbf{y},-\omega,-s}^+ \psi_{\mathbf{y},-\omega,-s}^- \\ V_4(\psi) = \frac{1}{2} \sum_{\omega,s} \int d\mathbf{x} d\mathbf{y} h(\mathbf{x} - \mathbf{y}) \psi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,s}^- \psi_{\mathbf{y},\omega,-s}^+ \psi_{\mathbf{y},\omega,-s}^- \quad (3.4)$$

where  $h(\mathbf{x} - \mathbf{y})$  is a rotational invariant potential, of the form

$$h(\mathbf{x} - \mathbf{y}) = \frac{1}{L^2} \sum_{\mathbf{p}} \hat{h}(\mathbf{p}) e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}, \quad (3.5)$$

with  $|\hat{h}(\mathbf{p})| \leq C e^{-\mu|\mathbf{p}|}$ , for some constants  $C$ ,  $\mu$ , and  $\hat{h}(0) = 1$ .

The model with  $g_{1,\perp} = 0$  is invariant under the global phase transformation  $\psi_{\mathbf{x},\omega,s}^\pm \rightarrow e^{\pm i\alpha_{\omega,s}} \psi_{\mathbf{x},\omega,s}^\pm$ , with the constant phase  $\alpha_{\omega,s}$  depending both on  $\omega$  and  $s$ . However, if  $g_{1,\perp} \neq 0$ , the model is only invariant under the transformation  $\psi_{\mathbf{x},\omega,s}^\pm \rightarrow e^{\pm i\alpha_\omega} \psi_{\mathbf{x},\omega,s}^\pm$  with the phase independent of  $s$ .

The removal of the ultraviolet cut-off is controlled by an easy extension of the analysis given in §2 of [36] or in §3 of [39] for a spinless model with interaction  $\lambda V_{\parallel}(\psi)$ ; the presence of the other terms (including the  $g_3$ -interaction considered in App. B) produces more lengthy expressions but introduces no extra difficulties. The crucial idea is to use an improvement respect to the power counting bounds due to the non-locality of the interaction, and to use that the "fermionic bubble" (see (2.39) of [36] or (3.17) of [28]) is exactly vanishing.

Regarding the removal of the infrared cut-off, we perform a multiscale analysis very similar to the one given in §2, that we shall sketch in App. B and App. C (we will refer to §4 of [24] for more details). It turns out that the infrared cut-off cannot be removed by this technique for all values of the couplings, but we are able to consider only two situations leading to a bounded flow:

1. the case  $g_{1,\perp} = 0$
2. the case  $g_{\parallel} = g_{\perp} - g_{1,\perp}$  and  $g_{1,\perp} > 0$

In the first case, when the ultraviolet and infrared cut-offs are removed, the model becomes exactly solvable, a property related to the invariance under the *local* phase transformation  $\psi_{\mathbf{x},\omega,s}^\pm \rightarrow e^{\pm i\alpha_{\mathbf{x},\omega,s}} \psi_{\mathbf{x},\omega,s}^\pm$ . Indeed, as we will see in §3.2–§3.5, the functional integrals generating the correlations can be exactly computed, up to corrections which are proved to be vanishing in the removed cutoffs limit. This will allow us to prove (2.45) and that the exponents of this model and the Hubbard model (also analyzed via functional integrals) are the same, if the parameters of the effective model are suitable chosen (see §3.6 below), so that the universal relations (1.22) follow. In the second case the model is not solvable, but still some correlations can be exactly computed, see §5, and this, again via a fine tuning of the effective model parameters, allows us to prove the relation (1.24).

### 3.2 Ward Identities in the $g_{1,\perp} = 0$ case

In the  $g_{1,\perp} = 0$  case we can derive a set of Ward Identities (WI). If we make in the generating functional (3.1) the gauge transformation  $\psi_{\mathbf{x},\omega,s}^\varepsilon \rightarrow e^{i\alpha_{\mathbf{x},\omega,s}} \psi_{\mathbf{x},\omega,s}^\varepsilon$ , we obtain, in the limit of removed cutoffs, the *functional Ward identity* (WI):

$$\begin{aligned} D_\mu(\mathbf{p}) \frac{\partial \mathcal{W}(J, \eta)}{\partial \hat{J}_{\mathbf{p},\mu,s}} - D_{-\mu}(\mathbf{p}) \sum_{\sigma,r} \nu_{sr}^{\mu\sigma}(\mathbf{p}) \frac{\partial \mathcal{W}(J, \eta)}{\partial \hat{J}_{\mathbf{p},-\sigma,r}} = \\ - D_{-\mu}(\mathbf{p}) \frac{\hat{J}_{-\mathbf{p},\mu,s}}{4\pi Z^2 c} + B_{\mathbf{p},\mu,s}(J, \eta) \end{aligned} \quad (3.6)$$

where  $D_\mu(\mathbf{p}) = -ip_0 + c\omega p$ ,

$$B_{\mathbf{p},\mu,s}(J, \eta) = \frac{1}{Z} \int \frac{d\mathbf{k}}{(2\pi)^2} \left[ \hat{\eta}_{\mathbf{k}+\mathbf{p},\mu,s}^+ \frac{\partial \mathcal{W}}{\partial \hat{\eta}_{\mathbf{k},\mu,s}^+} - \frac{\partial \mathcal{W}}{\partial \hat{\eta}_{\mathbf{k}+\mathbf{p},\mu,s}^-} \hat{\eta}_{\mathbf{k},\mu,s}^- \right]$$

and

$$\nu_s^\omega(\mathbf{p}) = [\delta_{\omega,1} (\delta_{s,-1}g_\perp + \delta_{s,1}g_\parallel) + \delta_{\omega,-1}\delta_{s,-1}g_4] \frac{\hat{h}(\mathbf{p})}{4\pi c} \quad (3.7)$$

Summing (3.6) over  $s$  we obtain the *charge Ward identity*:

$$\begin{aligned} & \left[ D_\mu(\mathbf{p}) - \nu_4(\mathbf{p})D_{-\mu}(\mathbf{p}) \right] \sum_s \frac{\partial \mathcal{W}}{\partial \hat{J}_{\mathbf{p},\mu,s}} - 2\nu_\rho(\mathbf{p})D_{-\mu}(\mathbf{p}) \sum_s \frac{\partial \mathcal{W}}{\partial \hat{J}_{\mathbf{p},-\mu,s}} \\ &= \sum_s \left[ -D_{-\mu}(\mathbf{p}) \frac{\hat{J}_{\mathbf{p},\mu,s}}{4\pi Z^2 c} + B_{\mathbf{p},\mu,s}(J, \eta) \right] \end{aligned} \quad (3.8)$$

with

$$\nu_4(\mathbf{p}) = g_4 \hat{h}(\mathbf{p})/(4\pi c), \quad 2\nu_\rho(\mathbf{p}) = (g_\parallel + g_\perp) \hat{h}(\mathbf{p})/(4\pi c)$$

Multiplying (3.6) by  $s$  and summing over  $s$  we obtain the *spin Ward identity*:

$$\begin{aligned} & \left[ D_\mu(\mathbf{p}) + \nu_4(\mathbf{p})D_{-\mu}(\mathbf{p}) \right] \sum_s s \frac{\partial \mathcal{W}}{\partial \hat{J}_{\mathbf{p},\mu,s}} - 2\nu_\sigma(\mathbf{p})D_{-\mu}(\mathbf{p}) \sum_s s \frac{\partial \mathcal{W}}{\partial \hat{J}_{\mathbf{p},-\mu,s}} \\ &= \sum_s s \left[ -D_{-\mu}(\mathbf{p}) \frac{\hat{J}_{\mathbf{p},\mu,s}}{4\pi Z^2 c} + B_{\mathbf{p},\mu,s}(J, \eta) \right] \end{aligned} \quad (3.9)$$

with

$$2\nu_\sigma(\mathbf{p}) = (g_\parallel - g_\perp) \hat{h}(\mathbf{p})/(4\pi c) \quad (3.10)$$

The proof of (3.8) and (3.9) with  $J = 0$  is essentially identical the one in §2 of [36] or in §3 of [39], while the presence of the term linear in  $J$  in the r.h.s. is explained in Sec. IV B of [40] (see also App. A of [27]); hence, we will omit here the details.

Note the presence, in the WI (3.8) and (3.9), of the  $\nu_\sigma$ ,  $\nu_\rho$ ,  $\nu_4$  terms, which are called *anomalies*; they appear as a consequence of the breaking of local symmetries in the functional integral (3.1), due the momentum cut-off  $\chi_{l,N}(\mathbf{k})$  in the fermionic integration. This symmetry breaking produces extra terms in the WI which do not vanish when the cut-offs are removed. Note also that such anomalies are linear in the coupling. Such a property is called *anomaly non renormalization*, and is crucially related to the non locality of the interaction [36]; in presence of local interactions, like in the massless Thirring model, it can be violated [38]. Another important point to be stressed is that (3.8) is true also when  $g_{1,\perp} > 0$ , while (3.9) is not; this remark will be used in the proof of Theorem 1.2.

By some other simple algebra we obtain from (3.13),(3.14) the identity

$$\frac{\partial \mathcal{W}}{\partial \hat{J}_{\mathbf{p},\mu',s'}} = \sum_{\mu,s} \frac{M_{\mu',\mu}^\rho(\mathbf{p}) + s' s M_{\mu',\mu}^\sigma(\mathbf{p})}{2} \left[ -D_{-\mu}(\mathbf{p}) \frac{\hat{J}_{\mathbf{p},\mu,s}}{4\pi Z^2 c} + B_{\mathbf{p},\mu,s}(J, \eta) \right] \quad (3.11)$$

where, if  $\gamma = \rho, \sigma$ , and setting  $\nu_{4,\rho} = -\nu_{4,\sigma} = \nu_4$ ,

$$\begin{aligned} M_{\mu',\mu}^\gamma(\mathbf{p}) &= \frac{\left[ D_{-\mu}(\mathbf{p}) - \nu_{4,\gamma}(\mathbf{p})D_\mu(\mathbf{p}) \right] \delta_{\mu',\mu} + \left[ 2\nu_\gamma(\mathbf{p})D_\mu(\mathbf{p}) \right] \delta_{\mu',-\mu}}{\left[ D_+(\mathbf{p}) - \nu_{4,\gamma}D_-(\mathbf{p}) \right] \left[ D_-(\mathbf{p}) - \nu_{4,\gamma}(\mathbf{p})D_+(\mathbf{p}) \right] - 4\nu_\gamma^2(\mathbf{p})D_+(\mathbf{p})D_-(\mathbf{p})} \\ &= \frac{u_{\gamma,+}(\mathbf{p})\delta_{\mu',\mu} + w_{\gamma,+}(\mathbf{p})\delta_{\mu',-\mu}}{-iv_{\gamma,+}(\mathbf{p})(p_0 + i\mu v_\gamma(\mathbf{p})cp_1)} + \frac{u_{\gamma,-}(\mathbf{p})\delta_{\mu',\mu} + w_{\gamma,-}(\mathbf{p})\delta_{\mu',-\mu}}{-iv_{\gamma,+}(\mathbf{p})(p_0 - i\mu v_\gamma(\mathbf{p})cp_1)} \end{aligned} \quad (3.12)$$

for

$$\begin{aligned} u_{\gamma,\mu}(\mathbf{p}) &= \frac{1}{2} \left[ \frac{1 - \nu_{4,\gamma}(\mathbf{p})}{v_{\gamma,+}(\mathbf{p})} + \mu \frac{1 + \nu_{4,\gamma}(\mathbf{p})}{v_{\gamma,-}(\mathbf{p})} \right] \\ w_{\gamma,\mu}(\mathbf{p}) &= \nu_\gamma(\mathbf{p}) \left[ \frac{1}{v_{\gamma,+}(\mathbf{p})} - \mu \frac{1}{v_{\gamma,-}(\mathbf{p})} \right] \end{aligned} \quad (3.13)$$

$$v_{\gamma,\mu}^2(\mathbf{p}) = \left(1 - \mu\nu_{4,\gamma}(\mathbf{p})\right)^2 - 4\nu_{\gamma}^2(\mathbf{p}), \quad v_{\gamma}(\mathbf{p}) = v_{\gamma,-}(\mathbf{p})/v_{\gamma,+}(\mathbf{p}) \quad (3.14)$$

By doing suitable functional derivatives of the functional WI (3.8), we can get many WI between the correlation functions. For example, if we make two derivatives w.r.t  $\widehat{\eta}_{\mathbf{p}+\mathbf{k},\omega,s}^+$  and  $\widehat{\eta}_{\mathbf{k},\omega,s}^-$  in both sides of (3.8), we sum over  $\mu$  and we put  $\eta = J = 0$ , we get the following *charge vertex Ward identity*:

$$\begin{aligned} -ip_0 \left[1 - 2\nu_{\rho}(\mathbf{p}) - \nu_4(\mathbf{p})\right] G_{\rho;\omega,s}(\mathbf{p}; \mathbf{p} + \mathbf{k}) + cp_1 \left[1 - 2\nu_{\rho}(\mathbf{p}) + \nu_4(\mathbf{p})\right] G_{j;\omega,s}(\mathbf{p}; \mathbf{p} + \mathbf{k}) \\ = \frac{1}{Z} [G_{2;\omega,s}(\mathbf{k}) - G_{2;\omega,s}(\mathbf{p} + \mathbf{k})] \end{aligned} \quad (3.15)$$

where

$$G_{\rho;\omega,s}(\mathbf{p}; \mathbf{p} + \mathbf{k}) = \sum_{\mu,t} \frac{\partial^3 \mathcal{W}}{\partial \widehat{J}_{\mathbf{p},\mu,t} \partial \widehat{\eta}_{\mathbf{p}+\mathbf{k},\omega,s}^+ \partial \widehat{\eta}_{\mathbf{k},\omega,s}^-} \quad (3.16)$$

$$G_{j;\omega,s}(\mathbf{p}; \mathbf{p} + \mathbf{k}) = \sum_{\mu,t} \mu \frac{\partial^3 \mathcal{W}}{\partial \widehat{J}_{\mathbf{p},\mu,t} \partial \widehat{\eta}_{\mathbf{p}+\mathbf{k},\omega,s}^+ \partial \widehat{\eta}_{\mathbf{k},\omega,s}^-} \quad (3.17)$$

$$G_{2;\omega,s} = \frac{\partial^2 \mathcal{W}}{\partial \widehat{\eta}_{\mathbf{k},\omega,s}^+ \partial \widehat{\eta}_{\mathbf{k},\omega,s}^-} \quad (3.18)$$

In a similar way, the functional WI (3.11) can be used to obtain a closed expression for the correlations of the density operator  $\rho_{\mathbf{x},\omega,s} = \psi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,s}^-$ . In fact, if we take (3.11) with  $\eta = 0$  and we perform a derivative w.r.t.  $\widehat{J}_{-\mathbf{p},\mu,s}$ , we obtain:

$$\langle \widehat{\rho}_{\mathbf{p},\omega',s'} \widehat{\rho}_{-\mathbf{p},\omega,s} \rangle_T = -D_{-\omega}(\mathbf{p}) \frac{\widehat{h}(\mathbf{p})}{4\pi Z^2 c} \frac{M_{\omega',\omega}^{\rho}(\mathbf{p}) + s's M_{\omega',\omega}^{\sigma}(\mathbf{p})}{2} \quad (3.19)$$

which implies that

$$\langle \rho_{\mathbf{x},\omega',s'} \rho_{\mathbf{y},\omega,s} \rangle_T = \frac{1}{2} \left[ G_{\omega',\omega}^{\rho}(\mathbf{x} - \mathbf{y}) + s's G_{\omega',\omega}^{\sigma}(\mathbf{x} - \mathbf{y}) \right] \quad (3.20)$$

where

$$G_{\omega',\omega}^{\gamma}(\mathbf{x} - \mathbf{y}) = \frac{1}{4\pi Z^2 c} \int \frac{d\mathbf{p}}{(2\pi)^2} e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})} \frac{p_0^2 + c^2 p^2}{D_{\omega}(\mathbf{p})} M_{\omega',\omega}^{\gamma}(\mathbf{p})$$

### 3.3 Schwinger-Dyson and Closed equations.

By substituting the Ward Identities found in the previous section in the Schwinger-Dyson equations, one get a set of closed equations, up to corrections which are vanishing in the limit of removed cut-offs; this is due to the non locality of the interaction and the proof is essentially identical to the one in [36] or in §4 of [39] for the spinless case. The presence of the spin makes however the resulting closed equations much more complex, and new properties emerge from their solution, like the spin-charge separation phenomenon.

Given any  $F(\psi)$  which is a power series in the field, the Wick Theorem says that, if  $\langle \cdot \rangle$  denotes the expectation w.r.t. the free measure,

$$\langle \widehat{\psi}_{\mathbf{k},\omega,s}^- F(\psi) \rangle_0 = \widehat{g}_{D,\omega}(\mathbf{k}) \langle \frac{\partial F(\psi)}{\partial \widehat{\psi}_{\mathbf{k},\omega,s}^+} \rangle_0. \quad (3.21)$$

It follows that

$$\frac{\partial e^{\mathcal{W}}}{\partial \widehat{\eta}_{\mathbf{k},\omega,s}^+}(0, \eta) = \langle \widehat{\psi}_{\mathbf{k},\omega,s}^- e^{\mathcal{V}(\psi,0,\eta)} \rangle_0 = \widehat{g}_{D,\omega}(\mathbf{k}) \langle \frac{\partial}{\partial \widehat{\eta}_{\mathbf{k},\omega,s}^+} e^{\mathcal{V}(\psi,\eta)} \rangle_0 \quad (3.22)$$

Hence, by using (3.3) and (3.7), we find, in the removed cutoffs limit:

$$D_\omega(\mathbf{k}) \frac{\partial e^\mathcal{W}}{\partial \widehat{\eta}_{\mathbf{k},\omega,s}^+}(0, \eta) = \frac{1}{Z} \widehat{\eta}_{\mathbf{k},\omega,s} e^{\mathcal{W}(0,\eta)} - Z \sum_{\omega',s'} \int \frac{d\mathbf{p}}{(2\pi)^2} 4\pi c \nu_{ss'}^{\omega\omega'}(\mathbf{p}) \frac{\partial^2 e^\mathcal{W}}{\partial \widehat{J}_{\mathbf{p},-\omega',s'} \partial \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+}(0, \eta) \quad (3.23)$$

that we shall call the *Schwinger-Dyson equation*. Let us now take the WI in the form (3.11), with  $J = 0$  and  $\mu' = -\omega'$ , derive it w.r.t.  $\widehat{\eta}_{\mathbf{p}+\mathbf{k},\omega,s}^+$  and insert the resulting expression in (3.23); we obtain the following closed equation

$$D_\omega(\mathbf{k}) \frac{\partial e^\mathcal{W}}{\partial \widehat{\eta}_{\mathbf{k},\omega,s}^+} = \frac{1}{Z} \widehat{\eta}_{\mathbf{k},\omega,s} e^\mathcal{W} - \sum_{\mu,t} \int \frac{d\mathbf{p} d\mathbf{q}}{(2\pi)^4} \widehat{F}_{-\omega,st}^{-\omega\mu}(\mathbf{p}) \cdot \left[ \widehat{\eta}_{\mathbf{q}+\mathbf{p},\mu,t}^+ \frac{\partial^2 e^\mathcal{W}}{\partial \widehat{\eta}_{\mathbf{q},\mu,t}^+ \partial \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+} - \frac{\partial^2 e^\mathcal{W}}{\partial \widehat{\eta}_{\mathbf{k}+\mathbf{p},\omega,s}^+ \partial \widehat{\eta}_{\mathbf{q}+\mathbf{p},\mu,t}^+} \widehat{\eta}_{\mathbf{q},\mu,t}^- \right], \quad (3.24)$$

where, if  $M_{\omega,\omega'}^\gamma$  is defined as in (3.12),

$$\widehat{F}_{\omega,s}^\mu(\mathbf{p}) = 4\pi c \sum_{\omega',s'} \nu_{ss'}^{\omega\omega'}(\mathbf{p}) M_{\omega',\omega\mu}^{s'}(\mathbf{p}), \quad M_{\mu,\mu'}^s = \frac{1}{2}(M_{\mu,\mu'}^\rho + s M_{\mu,\mu'}^\sigma)$$

### 3.4 The two point function

By using (3.24), we easily get:

$$\langle \psi_{\mathbf{x},\omega,s}^- \psi_{\mathbf{y},\omega',s'}^+ \rangle = \delta_{\omega,\omega'} \delta_{s,s'} S_\omega(\mathbf{x} - \mathbf{y})$$

where  $S_\omega(\mathbf{x})$  is the solution of the equation:

$$(\partial_\omega S_\omega)(\mathbf{x}) - F_{-\omega,+}^-(\mathbf{x}) S_\omega(\mathbf{x}) = \frac{1}{Z} \delta(\mathbf{x}), \quad (3.25)$$

with  $\partial_\omega = \partial_{x_0} + i\omega c \partial_{x_1}$ . The solution of (3.25) is:

$$S_\omega(\mathbf{x}) = e^{\Delta_+^-(\mathbf{x}|0)} g_\omega(\mathbf{x}), \quad g_\omega(\mathbf{x}) = \frac{1}{2\pi Z} \frac{1}{cx_0 + i\omega x}, \quad (3.26)$$

having defined  $\Delta_s^\varepsilon$  such that  $\partial_\omega^\mathbf{x} \Delta_s^\varepsilon(\mathbf{x}|\mathbf{z}) = F_{-\omega,s}^\varepsilon(\mathbf{x})$ :

$$\Delta_s^\varepsilon(\mathbf{x}|\mathbf{z}) = \int \frac{d\mathbf{k}}{(2\pi)^2} \frac{e^{-i\mathbf{k}\mathbf{x}} - e^{-i\mathbf{k}\mathbf{z}}}{D_\omega(\mathbf{k})} \widehat{F}_{-\omega,s}^\varepsilon(-\mathbf{k}) = \Delta_\rho^\varepsilon(\mathbf{x}|\mathbf{z}) + s \Delta_\sigma^\varepsilon(\mathbf{x}|\mathbf{z}). \quad (3.27)$$

for

$$\begin{aligned} \hat{\Delta}_\rho^\varepsilon(\mathbf{p}) &= g_\rho \hat{h}(-\mathbf{p}) \frac{M_{-\omega,-\omega\varepsilon}^\rho(-\mathbf{p})}{D_\omega(\mathbf{p})} + \frac{g_4}{2} \hat{h}(-\mathbf{p}) \frac{M_{\omega,-\omega\varepsilon}^\rho(-\mathbf{p})}{D_\omega(\mathbf{p})} \\ \hat{\Delta}_\sigma^\varepsilon(\mathbf{p}) &= g_\sigma \hat{h}(-\mathbf{p}) \frac{M_{-\omega,-\omega\varepsilon}^\sigma(-\mathbf{p})}{D_\omega(\mathbf{p})} - \frac{g_4}{2} \hat{h}(-\mathbf{p}) \frac{M_{\omega,-\omega\varepsilon}^\sigma(-\mathbf{p})}{D_\omega(\mathbf{p})} \end{aligned}$$

In order to evaluate the asymptotic behavior of  $\Delta_s^\varepsilon(\mathbf{x}|0)$ , we need to study functions of the type

$$I_{\omega,\varepsilon}(\mathbf{x}) = \int \frac{d^2\mathbf{p}}{(2\pi)^2} a(\mathbf{p}) \frac{e^{-i\mathbf{p}\cdot\mathbf{x}} - 1}{(p_0 + i\omega c p_1)[v_+(\mathbf{p})p_0 - i\varepsilon \omega v_-(\mathbf{p})c p_1]} \quad (3.28)$$

where  $a(\mathbf{p})$  and  $v_s(\mathbf{p}) > 0$  are even smooth functions of fast decrease. It is easy to show that

$$I_{\omega,\varepsilon}(\mathbf{x}) = \frac{a(0)}{v_+(0)} \tilde{I}_{\omega,\varepsilon}(\mathbf{x}) + A + O(1/|\mathbf{x}|) \quad (3.29)$$

where  $A$  is a *real* constant and, if  $v = v_-(0)/v_+(0)$ ,

$$\tilde{I}_{\omega,\varepsilon}(\mathbf{x}) = \int_{-1}^{+1} \frac{dp_1}{(2\pi c)} \int_{-\infty}^{+\infty} \frac{dp_0}{(2\pi)} \frac{e^{-i(p_0 x_0 + p_1 x_1/c)} - 1}{(p_0 + i\omega p_1)(p_0 - i\varepsilon \omega v p_1)}$$

One can see that, if  $v > 0$ ,  $v \neq 1$  and  $\mathbf{x} \neq 0$ ,

$$\tilde{I}_{\omega,\varepsilon}(\mathbf{x}) = \frac{1}{2\pi c(1+\varepsilon v)} [F(x_0, \omega x_1/c) + \varepsilon F(vx_0, -\varepsilon \omega x_1/c)]$$

where

$$F(x_0, x_1) = \int_0^1 \frac{dp_1}{p_1} \left[ e^{-p_1(|x_0| + i \operatorname{sgn}(x_0)x_1)} - 1 \right] = \ln |z| + i \operatorname{Arg}(\operatorname{sgn}(x_0)z) + B + O(1/z)$$

where  $z = x_0 + ix_1$ ,  $B$  is a real constant and  $|\operatorname{Arg}(z)| \leq \pi$ . Since

$$\operatorname{Arg}(\operatorname{sgn}(x_0)z) = \operatorname{Arg}(z) - \vartheta(x_0) \operatorname{sgn}(x_1)\pi$$

the function  $F(\mathbf{x})$  (considered only for  $|\mathbf{x}| > 1$ ) is discontinuous at  $x_0 = 0$ , while  $\tilde{I}_{\omega,\varepsilon}(\mathbf{x})$  is continuous. We can then write

$$\tilde{I}_{\omega,\varepsilon}(\mathbf{x}) = -\frac{1}{2\pi c(1+\varepsilon v)} [\log(x_0 + i\omega x_1/c) + \varepsilon \log(vx_0 - i\varepsilon \omega x_1/c)] + C + O(1/|\mathbf{x}|) \quad (3.30)$$

where  $C$  is again a real constant. By using (3.12), (3.28), (3.29) and (3.30), one can easily check that

$$\begin{aligned} \Delta_\gamma^\varepsilon(\mathbf{x}|0) &= -\frac{H_{\gamma,\varepsilon}^\varepsilon}{4\pi c} \ln(v_\gamma^2 x_0^2 + (x_1/c)^2) - \frac{H_{\gamma,-}^\varepsilon + H_{\gamma,+}^\varepsilon}{4\pi c} \ln \frac{x_0 + i\omega x_1/c}{v_\gamma x_0 + i\omega x_1/c} \\ &\quad + C_\gamma^\varepsilon + O(1/|\mathbf{x}|) \end{aligned} \quad (3.31)$$

for

$$\begin{aligned} H_{\gamma,\varepsilon}^+ &= \frac{2g_\gamma u_{\gamma,\varepsilon} + g_{4,\gamma} w_{\gamma,\varepsilon}}{v_{\gamma,+} + \varepsilon v_{\gamma,-}} = \frac{\varepsilon g_\gamma}{v_{\gamma,+} v_{\gamma,-}} \\ H_{\gamma,\varepsilon}^- &= \frac{2g_\gamma w_{\gamma,\varepsilon} + g_{4,\gamma} u_{\gamma,\varepsilon}}{v_{\gamma,+} - \varepsilon v_{\gamma,-}} = -\frac{4\pi\varepsilon}{2\nu_{\gamma,+}\nu_{\gamma,-}} \left[ 1 - \frac{(v_{\gamma,-} - \varepsilon v_{\gamma,+})^2}{4} \right] \end{aligned}$$

where  $C_\gamma^\pm$  are real constants and  $v_\gamma = v_{\gamma,+}(0)/v_{\gamma,-}(0)$  (and  $g_{4,\rho} = g_4$  while  $g_{4,\sigma} = -g_4$ ).

By using (3.13) and (3.14),

$$\frac{H_{\gamma,+}^+}{4\pi c} = \frac{\nu_\gamma}{v_{\gamma,+} v_{\gamma,-}} = \frac{\zeta_\gamma}{2} \quad \frac{H_{\gamma,-}^+ + H_{\gamma,+}^+}{4\pi c} = 0 \quad (3.32)$$

$$\frac{H_{\gamma,-}^-}{4\pi c} = \frac{1 - \frac{1}{4}(v_{\gamma,+} + v_{\gamma,-})^2}{2v_{\gamma,+} v_{\gamma,-}} = \frac{\eta_\gamma}{2} \quad \frac{H_{\gamma,-}^- + H_{\gamma,+}^-}{4\pi c} = -\frac{1}{2} \quad (3.33)$$

Note that this expression is continuous in  $v_\gamma = 1$ , as one expects, and that, at least at small coupling,  $\eta_\gamma \geq 0$ .



By using (3.26) and (3.27), we finally get

$$S_\omega(\mathbf{x}) = \frac{1}{2\pi Z} \frac{(c^2 v_\rho^2 x_0^2 + x_1^2)^{-\eta_\rho/2} (c^2 v_\sigma^2 x_0^2 + x_1^2)^{-\eta_\sigma/2}}{(c v_\rho x_0 + i\omega x_1)^{1/2} (c v_\sigma x_0 + i\omega x_1)^{1/2}} e^{C+O(1/|\mathbf{x}|)} \quad (3.34)$$

where  $C$  is a real constant  $O(g)$  and  $z^{1/2} = |z|^{1/2} e^{i \text{Arg}(z)/2}$ . Note that the leading term is well defined and continuous at any  $\mathbf{x} \neq 0$ .

Note also that, if  $g_4 = 0$ ,  $v_\rho = v_\sigma = 1$  and  $\eta_\rho = \eta_\sigma \equiv \eta/2$ , so that

$$S_\omega(\mathbf{x}) = \frac{1}{2\pi Z} \frac{(c^2 x_0^2 + x_1^2)^{-\eta/2}}{c x_0 + i\omega x_1} e^{C+O(1/|\mathbf{x}|)} \quad (3.35)$$

If we also put  $g_\perp = 0$  and  $g_\parallel = \lambda$ , we get for  $\eta$  the value found for the regularized Thirring model, that is  $\eta = 2\tau^2/(1 - \tau^2)$ , with  $\tau = \lambda/(4\pi c)$ ; see eq. (4.21) of [39].

### 3.5 The four point functions and the densities correlations

We want to calculate the truncated correlations  $\langle O_{\mathbf{x}}^{(t)} O_{\mathbf{y}}^{(t)} \rangle^T$  of the local quadratic operators  $O_{\mathbf{x}}^{(t)}$ ,  $t = (1, \alpha)$  or  $(2, \alpha)$ , defined as the analogous operators of the Hubbard model, see (2.28) and (2.29); note that  $p_F$  has no special meaning in the effective model, but it is left there since we want to compare the correlations in the two models.

Our UV regularization implies that  $\langle O_{\mathbf{x}}^{(t)} \rangle = 0$  for any  $t$ ; hence we can make the calculation very simply, by using the explicit expressions of the four points functions which follow from the closed equation (3.24) and then evaluating them so that the two coordinates corresponding to each  $O^{(t)}$  operator coincide, if this is meaningful. This works for all values of  $t$ , except  $(1, C)$  and  $(1, S_3)$ , where there is a singularity, related to the fact that the operators  $\rho_{\mathbf{x}, \omega, s} = \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, s}^-$  are not well defined in the limit  $N \rightarrow \infty$ , because of the singularity of the free propagator at  $\mathbf{x} = 0$ . However, in these cases we can use directly the WI (3.19) for the density correlations, which allows us to calculate correctly, in the limit  $N \rightarrow \infty$ , the correlations of  $O_{\mathbf{x}}^{(1, C)}$  and  $O_{\mathbf{x}}^{(1, S_3)}$ , by using (3.20), (3.12) and the equations (3.28), (3.29), (3.30). We get, for  $|\mathbf{x}| > 1$ ,

$$\begin{aligned} G_{\omega, \omega}^\gamma(\mathbf{x}) &\simeq \frac{1 - v_\gamma^2}{8\pi^2 c^2 Z^2} \left[ \frac{u_{\gamma, +}}{v_{\gamma, +} - v_{\gamma, -}} \frac{1}{(v_\gamma x_0 + i\omega x_1/c)^2} - \frac{u_{\gamma, -}}{v_{\gamma, +} + v_{\gamma, -}} \frac{1}{(v_\gamma x_0 - i\omega x_1/c)^2} \right] \\ G_{-\omega, \omega}^\gamma(\mathbf{x}) &\simeq \frac{1 - v_\gamma^2}{8\pi^2 c^2 Z^2} \left[ \frac{w_{\gamma, +}}{v_{\gamma, +} - v_{\gamma, -}} \frac{1}{(v_\gamma x_0 + i\omega x_1/c)^2} - \frac{w_{\gamma, -}}{v_{\gamma, +} + v_{\gamma, -}} \frac{1}{(v_\gamma x_0 - i\omega x_1/c)^2} \right] \end{aligned}$$

the corrections being of order  $1/|\mathbf{x}|^3$ . This implies that, for  $|\mathbf{x}| > 1$ ,

$$\langle O_{\mathbf{0}}^{(1, C)} O_{\mathbf{x}}^{(1, C)} \rangle^T = \frac{v_\rho^2(1 - \nu_4 + 2\nu_\rho) + (1 + \nu_4 - 2\nu_\rho)}{2\pi Z^2 c^2 v_{\rho, +} v_{\rho, -}} \frac{v_\rho^2 x_0^2 - x^2/c^2}{(v_\rho^2 x_0^2 + x^2/c^2)^2} + O(1/|\mathbf{x}|^3) \quad (3.36)$$

while  $\langle O_{\mathbf{0}}^{(1, S_3)} O_{\mathbf{x}}^{(1, S_3)} \rangle^T$  is obtained from this expression, by replacing  $\nu_4$  with  $-\nu_4$  and  $\nu_\rho$  with  $\nu_\sigma$  (hence also  $v_\rho$ ,  $v_{\rho, +}$  and  $v_{\rho, -}$  with  $v_\sigma$ ,  $v_{\sigma, +}$  and  $v_{\sigma, -}$ ).

One can see, see Lemma 4.1 of [41], that the same result could be obtained starting from the four point function, if we take the limit  $\varepsilon \rightarrow 0$  of the expression obtained by the substitution of the density operator  $\rho_{\mathbf{x}, \omega, s}$  with the regularization

$$\rho_{\mathbf{x}, \omega, s}^\varepsilon = \int d\mathbf{u} \delta_\varepsilon(\mathbf{u} - \mathbf{x}) \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{u}, \omega, s}^-$$

where  $\delta_\varepsilon(\mathbf{x})$  is a smooth approximation of the delta function, rotational invariant (in agreement with our UV regularization), whose support does not contain the point  $\mathbf{x} = 0$ .

In order to calculate the other correlations, we first note that the only four points functions different from zero are those defined by the equation

$$G_{s_1, s_2}^{\omega_1, \omega_2}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) = \langle \psi_{\mathbf{x}, \omega_1, s_1}^- \psi_{\mathbf{y}, \omega_2, s_2}^- \psi_{\mathbf{u}, \omega_2, s_2}^+ \psi_{\mathbf{v}, \omega_1, s_1}^+ \rangle$$

By (3.24),  $G_{s_1, s_2}^{\omega_1, \omega_2}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v})$  is the solution of the equation:

$$\begin{aligned} (\partial_{\omega_1}^{\mathbf{x}} G_{s_1, s_2}^{\omega_1, \omega_2})(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) &= \delta(\mathbf{x} - \mathbf{v}) S_{\omega_2}(\mathbf{y} - \mathbf{u}) - \delta_{\omega_1, \omega_2} \delta_{s_1, s_2} \delta(\mathbf{x} - \mathbf{u}) S_{\omega_1}(\mathbf{y} - \mathbf{v}) + \\ &\left[ -F_{-\omega_1, s_1 s_2}^{-\omega_1 \omega_2}(\mathbf{x} - \mathbf{y}) + F_{-\omega_1, s_1 s_2}^{-\omega_1 \omega_2}(\mathbf{x} - \mathbf{u}) + F_{-\omega_1, +}^{-}(\mathbf{x} - \mathbf{v}) \right] G_{s_1, s_2}^{\omega_1, \omega_2}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) \end{aligned} \quad (3.37)$$

For the two-points correlation of  $O_{\mathbf{x}}^{(2, \alpha)}$  we are interested in the case  $\omega_1 = -\omega_2 = \omega$ . For  $G_s^{\omega}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) = G_{s', s s'}^{\omega, -\omega}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v})$  we find

$$G_s^{\omega}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) = e^{-\left[ \Delta_s^+(\mathbf{x} - \mathbf{y} | \mathbf{v} - \mathbf{y}) - \Delta_s^+(\mathbf{x} - \mathbf{u}, \mathbf{v} - \mathbf{u}) \right]} S_{\omega}(\mathbf{x} - \mathbf{v}) S_{-\omega}(\mathbf{y} - \mathbf{u}). \quad (3.38)$$

Therefore, for  $\alpha = C, S_3$  we set  $\mathbf{x} = \mathbf{u}, \mathbf{y} = \mathbf{v}$  and  $s = +$ , while for  $\alpha = S_1, S_2$  we set  $s = -$ ; for  $TC_1, TC_3$  we set  $\mathbf{u} = \mathbf{v}, \mathbf{x} = \mathbf{y}$  and  $s = +$ ; while for  $TC_2, SC$  we set  $s = -$ .

For the two-points correlation of  $O_{\mathbf{x}}^{(1, \alpha)}$ ,  $\alpha \neq C, S_3$ , we are interested in the case  $\omega_1 = \omega_2 = \omega$ . If  $\bar{G}_s^{\omega}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) = G_{s', s s'}^{\omega, \omega}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v})$  we find

$$\begin{aligned} \bar{G}_s^{\omega}(\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v}) &= e^{-\left[ \Delta_s^-(\mathbf{x} - \mathbf{y} | \mathbf{v} - \mathbf{y}) - \Delta_s^-(\mathbf{x} - \mathbf{u}, \mathbf{v} - \mathbf{u}) \right]} S_{\omega}(\mathbf{x} - \mathbf{v}) S_{\omega}(\mathbf{y} - \mathbf{u}) \\ &- \delta_{s, +} e^{-\left[ \Delta_+^-(\mathbf{x} - \mathbf{y} | \mathbf{u} - \mathbf{y}) - \Delta_+^-(\mathbf{x} - \mathbf{v}, \mathbf{u} - \mathbf{v}) \right]} S_{\omega}(\mathbf{x} - \mathbf{u}) S_{\omega}(\mathbf{y} - \mathbf{v}). \end{aligned} \quad (3.39)$$

For  $\alpha = SC$  we set  $\mathbf{x} = \mathbf{y}, \mathbf{u} = \mathbf{v}$  and  $s = -$ ; for  $\alpha = S_1, S_2$  we set  $\mathbf{x} = \mathbf{u}, \mathbf{y} = \mathbf{v}$  and  $s = -$ ; for  $\alpha = S_3, C$  we set  $\mathbf{x} = \mathbf{u}, \mathbf{y} = \mathbf{v}$  and  $s = +$ .

Therefore, it is easy to see, by using (3.27) and (3.34), that, for  $|\mathbf{x}| > 1$ ,

$$\begin{aligned} \langle O_{\mathbf{0}}^{(2, \alpha)} O_{\mathbf{x}}^{(2, \alpha)} \rangle^T &= \frac{1}{\pi^2 Z^2 c^2} \frac{\cos(2p_F x)^{m_{\alpha}}}{(v_{\rho}^2 x_0^2 + x^2/c^2)^{x_{\rho, t}}} \frac{1}{(v_{\sigma}^2 x_0^2 + x^2/c^2)^{x_{\sigma, t}}} + O(1/|\mathbf{x}|^3), \quad \forall \alpha \\ \langle O_{\mathbf{0}}^{(1, SC)} O_{\mathbf{x}}^{(1, SC)} \rangle^T &= -\frac{1}{\pi^2 Z^2 c^2} \frac{\cos(2p_F x)}{(v_{\rho}^2 x_0^2 + x^2/c^2)^{2\eta_{\rho}}} \frac{v_{\rho}^2 x_0^2 - x^2/c^2}{(v_{\rho}^2 x_0^2 + x^2/c^2)^2} + O(1/|\mathbf{x}|^3) \\ \langle O_{\mathbf{0}}^{(1, \alpha)} O_{\mathbf{x}}^{(1, \alpha)} \rangle^T &= \frac{1}{\pi^2 Z^2 c^2} \frac{1}{(v_{\sigma}^2 x_0^2 + x^2/c^2)^{2\eta_{\sigma}}} \frac{v_{\sigma}^2 x_0^2 - x^2/c^2}{(v_{\sigma}^2 x_0^2 + x^2/c^2)^2} + O(1/|\mathbf{x}|^3), \quad \alpha = S_1, S_2 \end{aligned} \quad (3.40)$$

where  $m_{\alpha} = 1$ , if  $\alpha = C, S_i$ , while  $m_{\alpha} = 0$ , if  $\alpha = SC, TC_i$ , and

$$x_{\gamma, t} = \begin{cases} \eta_{\gamma} - \zeta_{\gamma} + 1/2 & t = (2, C), (2, S_3) \\ \eta_{\gamma} - s(\gamma)\zeta_{\gamma} + 1/2 & t = (2, S_1), (2, S_2) \\ \eta_{\gamma} + \zeta_{\gamma} + 1/2 & t = (2, TC_1), (2, TC_3) \\ \eta_{\gamma} + s(\gamma)\zeta_{\gamma} + 1/2 & t = (2, SC), (2, TC_2) \end{cases} \quad (3.41)$$

Let us now consider the special case  $g_{\sigma} = 0$  (i.e.  $\eta_{\sigma} = \zeta_{\sigma} = 0$ ), which we use as a effective model for the Hubbard model. In this case, the equations (3.40) imply that  $\langle O_{\mathbf{0}}^{(t)} O_{\mathbf{x}}^{(t)} \rangle$  decays, for  $|\mathbf{x}| \rightarrow \infty$ , as  $|\mathbf{x}|^{-2X_t}$ , with

$$2X_t = \begin{cases} 2 + 2\eta_{\rho} - 2\zeta_{\rho} & t = (2, C), (2, S_i) \\ 2 + 2\eta_{\rho} + 2\zeta_{\rho} & t = (2, SC), (2, TC_i) \\ 2 + 4\eta_{\rho} & t = (1, SC) \\ 2 & t = (1, C), (1, S_i) \end{cases} \quad (3.42)$$

Note that

$$\eta_\rho = -\frac{1}{2} + \frac{4 - v_{\rho+}^2 - v_{\rho-}^2}{4v_{\rho+}v_{\rho-}}, \quad \xi_\rho = \frac{2\nu_\rho}{v_{\rho+}v_{\rho-}} \quad (3.43)$$

Let us now define  $K = 2X_{2,C} - 1$  and  $\tilde{K} = 2X_{2,SC} - 1$ . By using (3.14), we see that

$$K = \frac{(1 - 2\nu_\rho)^2 - \nu_4^2}{v_{\rho+}v_{\rho-}} = \sqrt{\frac{(1 - \nu_4) - 2\nu_\rho}{(1 - \nu_4) + 2\nu_\rho}} \sqrt{\frac{(1 + \nu_4) - 2\nu_\rho}{(1 + \nu_4) + 2\nu_\rho}} \quad (3.44)$$

$$\tilde{K} = \frac{(1 + 2\nu_\rho)^2 - \nu_4^2}{v_{\rho+}v_{\rho-}} = K^{-1}$$

$$4\eta_\rho = K + \tilde{K} - 2$$

These equations imply that all the critical indices  $X_t$  and the parameter  $\eta_\rho$  can be expressed in terms of the single parameter  $K$ , only depending on  $g_2/c$  and  $g_4/c$ . In the following section we will show that the coupling of the model (3.1) can be chosen so that its exponents coincide with the Hubbard ones; then, by some simple algebra, one can check the validity of the scaling relations (1.22).

### 3.6 Fine tuning of the parameters of the effective model

Let us call  $\tilde{v}_h = (\tilde{g}_{2,h}, \tilde{g}_{4,h}, \tilde{\delta}_h)$ ,  $h \leq 0$ , the running coupling constants in the effective model with ultraviolet cutoff  $\gamma^N$  and parameters

$$g_{1,\perp} = 0, \quad g_{\parallel} = g_{\perp} = \tilde{g}_{2,N}, \quad g_4 = \tilde{g}_{4,N}, \quad \delta = \tilde{\delta}_N \quad (3.45)$$

so that, in particular,  $c = v_F(1 + \tilde{\delta}_N)$ , and put  $\tilde{v}_N = (\tilde{g}_{2,N}, \tilde{g}_{4,N}, \tilde{\delta}_N)$ . We call  $v_h = (g_{2,h}, g_{4,h}, \delta_h)$ ,  $h \leq 0$ , the analogous constants in the Hubbard model, while  $\vec{v}_h$  will be defined as in §2.3, that is  $\vec{v}_h = (v_h, g_{1,h}, \nu_h)$ . The analysis of the RG flow given in §2 and App. B implies that, for  $h \leq 0$ ,

$$\tilde{v}_{h-1} = \tilde{v}_h + \beta^{(0,h)}(\tilde{v}_h, \dots, \tilde{v}_0) + \tilde{r}^{(h)}(\tilde{v}_h, \dots, \tilde{v}_0, \tilde{v}_N) \quad (3.46)$$

$$v_{h-1} = v_h + \beta^{(0,h)}(v_h, \dots, v_0) + r^{(h)}(\vec{v}_h, \dots, \vec{v}_0, \lambda) \quad (3.47)$$

where  $\beta^{(0,h)}(\tilde{v}_h, \dots, \tilde{v}_0)$  is the beta function of the effective model with parameters (3.45), modified so that the endpoints have scale  $\leq 0$ . Note that  $\beta^{(0,h)}(v_h, \dots, v_0)$  is the function  $\beta^{(h)}(v_h, \dots, v_0)$  defined in (2.41), modified so that, in its tree expansion, no trees containing endpoints of type  $g_1$  appear and the space integrals are done in terms of continuous variables, instead of lattice variables (the difference is given by exponentially vanishing terms). The crucial bound (C.8) and the short memory property imply that  $|\tilde{r}^{(h)}(\tilde{v}_h, \dots, \tilde{v}_N)| \leq C[\max_{k \geq h} |\tilde{v}_k|]^2 \gamma^{\vartheta h}$ , while the analysis of §2.3 implies that  $r^{(h)}(\vec{v}_h, \dots, \vec{v}_0, \lambda)$  satisfies a bound similar to (2.51).

**Lemma 3.1** *Given the Hubbard model with coupling  $\lambda$  such that  $g_{1,0} \in D_{\varepsilon,\delta}$ , it is possible to choose  $\tilde{v}_N$  as analytic function of  $\lambda$ , so that*

$$\tilde{g}_2 = 2\lambda \left[ \hat{v}(0) - \frac{1}{2}\hat{v}(2p_F) \right] + O(\lambda^{3/2}), \quad \tilde{g}_4 = 2\lambda\hat{v}(0) + O(\lambda^2), \quad \tilde{\delta} = O(\lambda) \quad (3.48)$$

and, if  $\tilde{v}_h$  are the r.c.c. of the effective model with parameters satisfying (3.45), while  $v_h$  are the r.c.c. of the Hubbard model, then,  $\forall h \leq 0$ ,

$$|v_h - \tilde{v}_h| \leq C \frac{|g_{1,0}|}{1 + (a/2)|g_{1,0}| |h|} \quad (3.49)$$

Moreover, the r.c.c.  $\tilde{v}_h$  have a well definite limit as  $N \rightarrow +\infty$  and this limit still satisfies (3.49).

**Proof** - We have seen in the previous sections that the flows (3.46) and (3.47) have well defined limits  $\tilde{v}_{-\infty}$  and  $v_{-\infty}$ , as  $h \rightarrow -\infty$ , if the initial values are small enough and  $g_{1,0} \in D_{\varepsilon,\delta}$ . Moreover, the proof of this property for the flow (3.46) implies that  $\tilde{v}_{-\infty}$  is a smooth invertible function of  $\tilde{v}_N$ , such that  $\tilde{v}_{-\infty} = \tilde{v}_N + O(\tilde{v}_N^2)$ ; let us call  $\tilde{v}_N(\tilde{v}_{-\infty}) = \tilde{v}_{-\infty} + O(\tilde{v}_{-\infty}^2)$  its inverse. It is also clear that  $\tilde{v}_N(\tilde{v}_{-\infty})$  has a well defined limit as  $N \rightarrow \infty$ , that we shall call  $\tilde{v}(\lambda)$ , and that this is true also for the r.c.c.  $\tilde{v}_h$ ,  $h \leq 0$ .

The previous remarks, together with (2.73) and (2.74), imply that it is possible to choose  $\tilde{v}_N$ , satisfying (3.48), so that

$$\tilde{v}_{-\infty} - v_{-\infty} = 0 \quad (3.50)$$

In order to prove (3.49), we note that, because of the bound (C.8) and the short memory property, in the effective model with couplings satisfying (3.48),

$$|\tilde{v}_h - \tilde{v}_{-\infty}| \leq C\lambda^2\gamma^{\vartheta h} \quad (3.51)$$

On the other hand, from Lemma 2.4 and 2.5

$$|v_h - v_{-\infty}| \leq C \sum_{j=-\infty}^h [|g_{1,j}|^2 + \lambda\gamma^{\frac{\vartheta}{2}j}] \leq C_1 \frac{|g_{1,0}|}{1 + (a/2)|g_{1,0}||h|} \quad (3.52)$$

These two bounds immediately imply (3.49). ■

Let us now note that the critical indices of the effective model can be calculated in terms of  $\tilde{v}_{-\infty}$  by the same procedure used for the Hubbard model in §2.4 and that we get an equation like (2.86), *with the same function*  $\beta_t^{(0,j)}$ . Hence, the above lemma allows us to conclude that the critical indices in the Hubbard model and in the effective model coincide, provided that the value of  $\tilde{v} = \lim_{N \rightarrow \infty} \tilde{v}_N$  is chosen properly. It follows that all the indices are given by the equations (3.42), with

$$\begin{aligned} \nu_\rho &= \frac{\tilde{g}_2(\lambda)}{4\pi c} = \lambda \frac{\hat{v}(0) - \hat{v}(2p_F)/2}{2\pi \sin \bar{p}_F} + O(\lambda^{3/2}) \\ \nu_4 &= \frac{\tilde{g}_4(\lambda)}{4\pi c} = \lambda \frac{\hat{v}(0)}{2\pi \sin \bar{p}_F} + O(\lambda^2) \end{aligned} \quad (3.53)$$

where (3.48) has been used, together with  $c = \sin \bar{p}_F + O(\lambda)$ . Moreover, (3.53) and (3.44) imply that  $K = 1 - 2\lambda[\hat{v}(0) - \hat{v}(2p_F)/2]/(\pi \sin \bar{p}_F) + O(\lambda^{3/2})$ , in agreement with (1.21).

## 4 Spin-Charge Separation

If  $\mathbf{k} \neq 0$ , the Fourier transform  $\hat{S}_2(\mathbf{k} + \omega \mathbf{p}_F)$  of the two-point Schwinger function  $S_2(\mathbf{x})$  in the Hubbard model can be written as a tree expansion, in a way similar to eq. (2.64) of [38], whom we shall refer to for the notation:

$$\hat{S}_2(\mathbf{k} + \mathbf{p}_F^\omega) = \sum_{n=0}^{\infty} \sum_{j_0=-\infty}^0 \sum_{\tau \in \mathcal{T}_{j_0,n,2,0}} \sum_{\substack{\mathbf{P} \in \mathcal{P} \\ |P_{v_0}|=2}} \hat{G}_{\tau,\omega}^2(\mathbf{k}) \quad (4.1)$$

where  $\mathbf{p}_F^\omega = (\omega p_F, 0)$ ,  $\omega = \pm$ . Here  $\hat{G}_{\tau,\omega}^2(\mathbf{k})$  represents the contribution of a single tree  $\tau$  with  $n$  endpoints and root of scale  $j_0$ ; if  $|\mathbf{k}| \in [\gamma^{h_{\mathbf{k}}}, \gamma^{h_{\mathbf{k}}+1})$ , it obeys the bound:

$$|\hat{G}_{\tau,\omega}^2(\mathbf{k})| \leq C\gamma^{-(h_{\mathbf{k}}-j_0)} \frac{\gamma^{-h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}} \sum_{n=0}^{\infty} \sum_{j_0=-\infty}^0 \sum_{\tau \in \mathcal{T}_{j_0,n,\mathbf{k}}} \sum_{\substack{\mathbf{P} \in \mathcal{P} \\ |P_{v_0}|=2}} (C|\lambda|)^n \prod_{v \text{ not e.p.}} \gamma^{-d_v} \frac{Z_{h_v}}{Z_{h_v-1}}, \quad (4.2)$$

where  $\mathcal{T}_{j_0, n, \mathbf{k}}$  denotes the family of trees whose special vertices (those associated with the external lines) have scale  $h_{\mathbf{k}}$  or  $h_{\mathbf{k}+1}$ . Moreover,  $d_v > 0$ , except for the vertices belonging to the path connecting the root with  $v^*$ , the higher vertex (of scale  $h^*$ ) preceding both the two special endpoints, where  $d_v$  can be equal to 0. These vertices can be regularized by using a factor  $\gamma^{-(h^*-j_0)}$ , extracted from the factor  $\gamma^{-(h_{\mathbf{k}}-j_0)}$ , so that we can safely perform the sum over all the trees with a fixed value of  $h^*$  and we get

$$|\hat{S}_2(\mathbf{k} + \mathbf{p}_F^\omega)| \leq C \frac{\gamma^{-h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}} \sum_{h^*=-\infty}^{h_{\mathbf{k}}} \gamma^{-(h_{\mathbf{k}}-h^*)} \leq C \frac{\gamma^{-h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}} \quad (4.3)$$

A similar bounds can be obtained for the effective model with  $g_{1\perp} = 0$  and couplings chosen as in Lemma 3.1. We shall call  $\hat{S}_\omega^M(\mathbf{k})$  and  $\tilde{Z}_h$  the two-point function Fourier transform and the renormalization constants, respectively, in this model.

Let us put

$$\hat{S}_2(\mathbf{k} + \mathbf{p}_F^\omega) = \frac{1}{Z_{h_{\mathbf{k}}}} \bar{G}_\omega^2(\mathbf{k}), \quad \hat{S}_\omega^M(\mathbf{k}) = \frac{1}{\tilde{Z}_{h_{\mathbf{k}}}} \bar{G}_\omega^{2,M}(\mathbf{k}) \quad (4.4)$$

We can write

$$\hat{S}_2(\mathbf{k} + \mathbf{p}_F^\omega) = \frac{\tilde{Z}_{h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}} \frac{\bar{G}_\omega^2(\mathbf{k})}{\tilde{Z}_{h_{\mathbf{k}}}} = \frac{\tilde{Z}_{h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}} \hat{S}_\omega^M(\mathbf{k}) + \frac{1}{Z_{h_{\mathbf{k}}}} [\bar{G}_\omega^2(\mathbf{k}) - \bar{G}_\omega^{2,M}(\mathbf{k})] \quad (4.5)$$

Note now that  $\bar{G}_\omega^2(\mathbf{k})$  differs from  $\bar{G}_\omega^{2,M}(\mathbf{k})$  for three reasons:

- 1) the propagators are different, which produces a difference exponentially small thanks to (2.39), (2.40) and the short memory property;
- 2) the r.c.c.  $v_h$  and  $\tilde{v}_h$  are different, which produces a difference of order  $\tilde{g}_{1,h_{\mathbf{k}}}$ , thanks to (3.49) and the short memory property;
- 3) in the tree expansion of  $\bar{G}_\omega^2(\mathbf{k})$  and of the ratios  $Z_j/Z_{j-1}$  there are trees with endpoints of type  $g_1$ , not present in the tree expansion of  $\bar{G}_\omega^{2,M}(\mathbf{k})$  and  $\tilde{Z}_j/\tilde{Z}_{j-1}$ ; this fact produces again a difference of order  $\tilde{g}_{1,h_{\mathbf{k}}}$ .

These remarks, together with the fact that there is no tree with only one endpoint in the tree expansion, implies that

$$\left| \frac{1}{Z_{h_{\mathbf{k}}}} [\bar{G}_\omega^2(\mathbf{k}) - \bar{G}_\omega^{2,M}(\mathbf{k})] \right| \leq C |\lambda \tilde{g}_{1,h_{\mathbf{k}}}| \frac{\gamma^{-h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}} \quad (4.6)$$

For similar reason, we have

$$\frac{\tilde{Z}_{h_{\mathbf{k}}}}{Z_{h_{\mathbf{k}}}} = \prod_{j=h_{\mathbf{k}}}^0 \frac{\tilde{Z}_{h_j}}{Z_{h_{j-1}}} \frac{\tilde{Z}_{h_0}}{Z_{h_0}} = [1 + O(\lambda^2)] e^{O(\lambda) \sum_{j=h_{\mathbf{k}}}^0 \tilde{g}_{1,j}} = [1 + O(\lambda)] L(|\mathbf{k}|^{-1})^{O(\lambda)} \quad (4.7)$$

where  $L(t)$ ,  $t \geq 1$ , is the same function defined in Theorem 1.1.

Theorem 1.3 easily follows from (4.6), (4.7) and the explicit expression (3.34) of  $S_\omega(\mathbf{x})$  in the effective model, applied to the case  $g_\sigma = 0$ ,  $c = v_F(1 + \tilde{\delta})$ .

## 5 Susceptibility and Drude weight

The effective model is not invariant under a gauge transformation with the phase depending both on  $\omega$  and  $s$ , if  $g_{1,\perp} > 0$ ; however, it is still invariant under a gauge transformation with the

phase only depending on  $\omega$ . This is true, in particular, if the interaction is spin symmetric, that is if  $g_{\parallel} = g_{\perp} - g_{1,\perp}$ , see item d in App. B. Since also the Hubbard model is spin symmetric, it is natural to see if one can use this “restricted” gauge invariance to get some useful information on the asymptotic behavior of the Hubbard model, by comparing it with the effective model with  $g_{1,\perp} > 0$ .

Let us put  $g_{\perp} \equiv \bar{g}_2$ ,  $g_{1\perp} \equiv \bar{g}_1$  and  $g_{\parallel} = \bar{g}_2 - \bar{g}_1$ . We want to show that we can choose the parameters of the effective model  $\bar{g}_1, \bar{g}_2, \bar{g}_4, \bar{\delta}$ , so that the running coupling constants are asymptotically close to those of the Hubbard model. This result is stronger of the similar one contained in Lemma 3.1, since now all the running couplings are involved, and this implies also that the values of  $\bar{g}_2, \bar{g}_4$  and  $\bar{\delta}$  are *different* with respect to the analogous constants defined in Lemma 3.1. The main consequence of these considerations is that we can use the restricted WI of this new effective model to get non trivial information on some Hubbard model correlation functions, not plagued by logarithmic corrections.

Let  $\vec{l}_h = (\bar{g}_{1,h}, \bar{g}_{2,h}, \bar{g}_{4,h}, \bar{\delta}_h)$ ,  $h \leq 0$ , be the running coupling constants appearing in the integration of the infrared part of the effective model. The smoothness properties of the integration procedure imply that, in the UV limit,  $\vec{l}_0$  is a smooth invertible function of the interaction parameters  $\vec{l} = (\bar{g}_1, \bar{g}_2, \bar{g}_4, \bar{\delta})$ , whose inverse we shall call  $\vec{l}(\vec{l}_0)$ ; hence we can fix the effective model by giving the value of  $\vec{l}_0$  and by putting  $\vec{l} = \vec{l}(\vec{l}_0)$ . In a similar way we call  $\vec{g}_h = (g_{1,h}, g_{2,h}, g_{4,h}, \delta_h)$ ,  $h \leq 0$ , the running couplings of the Hubbard model with coupling  $\lambda$ .

We now define  $\vec{x}_h = \vec{l}_h - \vec{g}_h$ ,  $h \leq 0$ . By using the decomposition (2.41) for  $\vec{g}_h$  and the similar one for  $\vec{l}_h$ , we can write

$$\vec{x}_{h-1} = \vec{x}_h + [\vec{\beta}_h^{(1)}(\vec{g}_h, \dots, \vec{g}_0) - \vec{\beta}_h^{(1)}(\vec{l}_h, \dots, \vec{l}_0)] + \vec{\beta}_h^{(2)}(\vec{g}_h, \nu_h \dots \vec{g}_0, \nu_0, \lambda) + \vec{\beta}_h^{(3)}(\vec{l}_h, \dots, \vec{l}_0, \vec{l}) \quad (5.1)$$

where  $\vec{\beta}_h^{(1)}$  coincides with the function  $\beta^{(h)}$  defined in (2.41). In the usual way, one can see that

$$|\vec{\beta}_h^{(1)}(\vec{g}_h, \dots, \vec{g}_0) - \vec{\beta}_h^{(1)}(\vec{l}_h, \dots, \vec{l}_0)| \leq C \left[ |\lambda| + \sup_{k \geq h} |\vec{l}_k| \right] \sum_{k=h}^0 \gamma^{-\vartheta(k-h)} |\vec{x}_k| \quad (5.2)$$

and that  $|\vec{\beta}_h^{(2)}| \leq C|\lambda|\gamma^{\vartheta h}$ ,  $|\vec{\beta}_h^{(3)}| \leq C[\sup_{k \geq h} |\vec{l}_k|]^2$ . Note that the different power in the coupling of these two bounds is due to the terms linear in  $\lambda$  in the beta function for  $\delta_h$ , which are present in the Hubbard model, while similar terms are absent in the effective model, see remark after (B.9) in App. B.

We want to show that, given  $\lambda$  positive and small enough, it is possible to choose  $\vec{l}_0$ , hence  $\vec{x}_0$ , so that  $\vec{x}_{-\infty} = 0$ ; we shall do that by a simple fixed point argument. Note that  $\vec{x}_{-\infty} = 0$  if and only if

$$\vec{x}_h = - \sum_{h=-\infty}^{\bar{h}} \{ [\vec{\beta}_h^{(1)}(\vec{g}_h, \dots, \vec{g}_0) - \vec{\beta}_h^{(1)}(\vec{l}_h, \dots, \vec{l}_0)] + \vec{\beta}_h^{(2)} + \vec{\beta}_h^{(3)} \} \quad (5.3)$$

We consider the Banach space  $\mathcal{M}_{\vartheta}$ ,  $\vartheta < 1$ , of sequences  $\vec{x} = \{\vec{x}_h\}_{h \leq 0}$  with norm  $\|\vec{x}\| = \sup_{k \leq 0} |\vec{x}_k| \gamma^{-(\vartheta/2)k}$  and the operator  $\mathbf{T} : \mathcal{M}_{\vartheta} \rightarrow \mathcal{M}_{\vartheta}$ , defined as the r.h.s. of (5.3). Given  $\xi > 0$ , let  $\mathcal{B}_{\xi} = \{\vec{x} \in \mathcal{M}_{\vartheta} : \|\vec{x}\| \leq \xi\}$ ; if  $\lambda$  is small enough, say  $\lambda \leq \varepsilon_0$  and  $\xi \leq \varepsilon_0$ , and  $\vec{l}_h = \vec{g}_h + \vec{x}_h$ , the functions  $\vec{\beta}_h^{(1)}(\vec{l}_h, \dots, \vec{l}_0)$  and  $\vec{\beta}_h^{(3)}(\vec{l}_h, \dots, \vec{l}_0, \vec{l})$  are well defined and satisfy the bounds above, even if  $\vec{x}$  is not the flow of the effective model corresponding to  $\vec{l}_0$ . Hence, we have:

$$\gamma^{-(\vartheta/2)h} |\mathbf{T}(\vec{x})_h| \leq c_0 \lambda (\xi \lambda + 1) \sum_{k=-\infty}^{h-1} \gamma^{\frac{\vartheta}{2}k} \leq c_1 \lambda (1 + \xi \lambda) \quad (5.4)$$

so that  $\mathcal{B}_\xi$  is invariant if  $\xi = 2c_1$  and  $\lambda \leq \varepsilon_1 = \min\{\varepsilon_0, \varepsilon_0/(2c_1), 1/(2c_1)\}$ . Moreover

$$\begin{aligned} \mathbf{T}(\vec{x})_h - \mathbf{T}(\vec{x}')_h &= \sum_{h=-\infty}^{\bar{h}} \{ [\vec{\beta}_h^{(1)}(\{\vec{g}_k + \vec{x}_k\}_{k \geq h}) - \vec{\beta}_h^{(1)}(\{\vec{g}_k + \vec{x}'_k\}_{k \geq h})] \\ &\quad + [\vec{\beta}_h^{(3)}(\{\vec{g}_k + \vec{x}'_k\}_{k \geq h}) - \vec{\beta}_h^{(3)}(\{\vec{g}_k + \vec{x}_k\}_{k \geq h})] \} \end{aligned}$$

and  $|\mathbf{T}(\vec{x})_h - \mathbf{T}(\vec{x}')_h| \leq c_2 \lambda \|x - x'\|$ , thanks to the fact that all the terms in the r.h.s. of this equation are of the second order in the running couplings. It follows that, if  $c_2 \lambda < 1$ ,  $\mathbf{T}$  is a contraction in  $\mathcal{B}_\xi$ , so that (5.3) has a unique solution  $\vec{x}^{(0)}$  in this set; moreover, if we put  $\vec{l}_h = \vec{g}_h + \vec{x}_h^{(0)}$ ,  $\{\vec{l}_h\}_{h \leq 0}$  is the flow of the effective model corresponding to a value of  $\vec{l}$  such that

$$|\vec{g}_h - \vec{l}_h| \leq C|\lambda| \gamma^{\frac{\vartheta}{2}h} \quad (5.5)$$

Finally, this solution is such that  $\vec{l}$  is equal to  $\vec{g}_0$  at the first order; hence, by using (2.26), we get

$$\bar{g}_1 = 2\lambda \hat{v}(2p_F) + O(\lambda^2), \quad \bar{g}_2 = 2\lambda \hat{v}(0) + O(\lambda^2), \quad \bar{g}_4 = 2\lambda \hat{v}(0) + O(\lambda^2), \quad \bar{\delta} = O(\lambda) \quad (5.6)$$

Thanks to the bound (5.5), this choice of  $\vec{l}$ , allows us to extend to the Hubbard model Lemma 1 of [27], proved for the spinless fermion model. Hence, we can say that there are constants  $Z = 1 + O(\lambda^2)$ ,  $Z_3 = 1 + O(\lambda)$  and  $\tilde{Z}_3 = v_F + O(\lambda)$  such that, if  $\kappa \leq 1$  and  $|\mathbf{p}| \leq \kappa$ ,

$$\begin{aligned} \hat{\Omega}_C(\mathbf{p}) &= Z_3^2 \langle \hat{\rho}_{\mathbf{p}} \hat{\rho}_{-\mathbf{p}} \rangle^{(g)} + A_c + o(\mathbf{p}) \\ \hat{D}(\mathbf{p}) &= -\tilde{Z}_3^2 \langle \hat{j}_{\mathbf{p}} \hat{j}_{-\mathbf{p}} \rangle^{(g)} + A_j + o(\mathbf{p}) \end{aligned} \quad (5.7)$$

where  $\langle \cdot \rangle^{(g)}$  denotes the expectation in the effective model satisfying (5.5),  $A_c$  and  $A_j$  are suitable  $O(1)$  constants and

$$\rho_{\mathbf{x}} = \sum_{\omega, s} \psi_{\mathbf{x}, \omega s}^+ \psi_{\mathbf{x}, \omega s} \quad j_{\mathbf{x}} = \sum_{\omega, s} \omega \psi_{\mathbf{x}, \omega s}^+ \psi_{\mathbf{x}, \omega s} \quad (5.8)$$

Moreover, if we put  $\mathbf{p}_F^\omega = (\omega p_F, 0)$  and we suppose that  $0 < \kappa \leq |\mathbf{p}|, |\mathbf{k}'|, |\mathbf{k}' - \mathbf{p}| \leq 2\kappa$ ,  $0 < \vartheta < 1$ , then

$$\begin{aligned} \hat{G}_\rho^{2,1}(\mathbf{k}' + \mathbf{p}_F^\omega, \mathbf{k}' + \mathbf{p} + \mathbf{p}_F^\omega) &= Z_3 \langle \hat{\rho}_{\mathbf{p}} \psi_{\mathbf{k}', \omega} \psi_{\mathbf{k}' + \mathbf{p}, \omega}^+ \rangle^{(g)} [1 + O(\kappa^\vartheta)] \\ \hat{G}_j^{2,1}(\mathbf{k}' + \mathbf{p}_F^\omega, \mathbf{k}' + \mathbf{p} + \mathbf{p}_F^\omega) &= \tilde{Z}_3 \langle \hat{j}_{\mathbf{p}} \psi_{\mathbf{k}', \omega} \psi_{\mathbf{k}' + \mathbf{p}, \omega}^+ \rangle^{(g)} [1 + O(\kappa^\vartheta)] \\ \hat{S}_2(\mathbf{k}' + \mathbf{p}_F^\omega) &= \langle \psi_{\mathbf{k}', \omega, \sigma}^- \psi_{\mathbf{k}, \omega, \sigma}^+ \rangle^{(g)} [1 + O(\kappa^\vartheta)] \end{aligned} \quad (5.9)$$

where  $G_\rho^{2,1}(\mathbf{x})$  and  $G_j^{2,1}(\mathbf{x})$  are defined after (1.13), while the functions  $\langle \hat{\rho}_{\mathbf{p}} \psi_{\mathbf{k}', \omega} \psi_{\mathbf{k}' + \mathbf{p}, \omega}^+ \rangle^{(g)}$  and  $\langle \hat{j}_{\mathbf{p}} \psi_{\mathbf{k}', \omega} \psi_{\mathbf{k}' + \mathbf{p}, \omega}^+ \rangle^{(g)}$  coincide with the functions (3.16) and (3.17), respectively, with  $c = v_F(1 + \bar{\delta})$ . As already mentioned, if  $g_1 > 0$ , the effective model is still invariant under a spin-independent phase transformation; hence the WI (3.6) is satisfied, if we sum both sides over  $s$  and we substitute  $\nu_s^\mu(\mathbf{p})$  with

$$\bar{\nu}_s^\mu(\mathbf{p}) = \{\delta_{\omega,1}[\delta_{s,-1}\bar{g}_2 + \delta_{s,1}(\bar{g}_2 - \bar{g}_1)] + \delta_{\omega,-1}\delta_{s,-1}\bar{g}_4\} \frac{\hat{h}(\mathbf{p})}{4\pi\bar{c}}, \quad \bar{c} = v_F(1 + \bar{\delta}) \quad (5.10)$$

Therefore we get the a WI similar to (3.15), that is

$$\begin{aligned} &-ip_0[1 - \bar{\nu}_4(\mathbf{p}) - 2\bar{\nu}_\rho(\mathbf{p})] \langle \hat{\rho}_{\mathbf{p}} \psi_{\mathbf{k}', \omega} \psi_{\mathbf{k}' + \mathbf{p}, \omega}^+ \rangle^{(g)} + \bar{c}p[1 + \bar{\nu}_4(\mathbf{p}) - 2\bar{\nu}_\rho(\mathbf{p})] \langle \hat{j}_{\mathbf{p}} \psi_{\mathbf{k}', \omega} \psi_{\mathbf{k}' + \mathbf{p}, \omega}^+ \rangle^{(g)} \\ &= \frac{1}{Z} \left[ \langle \psi_{\mathbf{k}, \omega, \sigma}^- \psi_{\mathbf{k}, \omega, \sigma}^- \rangle^{(g)} - \langle \psi_{\mathbf{k} + \mathbf{p}, \omega, \sigma}^- \psi_{\mathbf{k} + \mathbf{p}, \omega, \sigma}^- \rangle^{(g)} \right] \end{aligned} \quad (5.11)$$

where

$$\bar{\nu}_4(\mathbf{p}) = \bar{g}_4 \frac{\hat{h}(\mathbf{p})}{4\pi\bar{c}}, \quad \bar{\nu}_\rho(\mathbf{p}) = \frac{\bar{g}_2 - \bar{g}_1/2}{4\pi\bar{c}} \hat{h}(\mathbf{p}) \quad (5.12)$$

By replacing (5.9) in (5.11), and comparing with (1.14) we get, if  $\bar{\nu}_4(0) \equiv \bar{\nu}_4$ ,  $\bar{\nu}_\rho(0) = \bar{\nu}_\rho$

$$\frac{Z_3}{Z} = (1 - \bar{\nu}_4 - 2\bar{\nu}_\rho), \quad \frac{\tilde{Z}_3}{Z} = \bar{c}(1 + \bar{\nu}_4 - 2\bar{\nu}_\rho) \quad (5.13)$$

Moreover, by proceeding as in derivation of (3.19), we get:

$$\begin{aligned} D_\omega(\mathbf{p}) \langle \rho_{\mathbf{p},\omega}^{(c)} \rho_{-\mathbf{p},\omega'}^{(c)} \rangle^{(g)} - \bar{\nu}_4(\mathbf{p}) D_{-\omega}(\mathbf{p}) \langle \rho_{\mathbf{p},\omega}^{(c)} \rho_{-\mathbf{p},\omega'}^{(c)} \rangle^{(g)} \\ - 2\bar{\nu}_\rho(\mathbf{p}) D_{-\omega}(\mathbf{p}) \langle \rho_{\mathbf{p},-\omega}^{(c)} \rho_{-\mathbf{p},\omega'}^{(c)} \rangle^{(g)} = -\delta_{\omega,\omega'} \frac{D_{-\omega}(\mathbf{p})}{2\pi Z^2 \bar{c}} \end{aligned} \quad (5.14)$$

Hence, by some simple algebra, we get:

$$\langle \rho_{\mathbf{p},\omega}^{(c)} \rho_{-\mathbf{p},\omega}^{(c)} \rangle^{(g)} = \frac{1}{2\pi Z^2 \bar{c} \bar{v}_{\rho,+}^2} \frac{D_{-\omega}(\mathbf{p}) [D_{-\omega}(\mathbf{p}) - \bar{\nu}_4 D_\omega(\mathbf{p})]}{p_0^2 + \bar{c}^2 \bar{v}_\rho^2 p^2} + O(\mathbf{p}) \quad (5.15)$$

$$\langle \rho_{\mathbf{p},\omega}^{(c)} \rho_{-\mathbf{p},-\omega}^{(c)} \rangle^{(g)} = \frac{1}{2\pi Z^2 \bar{c} \bar{v}_{\rho,+}^2} \frac{2\bar{\nu}_\rho D_\omega(\mathbf{p}) D_{-\omega}(\mathbf{p})}{p_0^2 + \bar{c}^2 \bar{v}_\rho^2 p^2} + O(\mathbf{p}) \quad (5.16)$$

where

$$\bar{v}_\rho = \bar{v}_{\rho,-}/v_{\rho,+}, \quad \bar{v}_{\rho,\mu}^2 = (1 - \mu \bar{\nu}_4)^2 - 4\bar{\nu}_\rho^2 \quad (5.17)$$

Therefore the charge and current density correlations are given by:

$$\langle \rho_{\mathbf{p}}^{(c)} \rho_{-\mathbf{p}}^{(c)} \rangle^{(g)} = \frac{1}{\pi Z^2 \bar{c} \bar{v}_{\rho,+}^2} \frac{-p_0^2(1 - \bar{\nu}_4 + 2\bar{\nu}_\rho) + \bar{c}^2 p^2(1 + \bar{\nu}_4 - 2\bar{\nu}_\rho)}{p_0^2 + \bar{c}^2 \bar{v}_\rho^2 p^2} + O(\mathbf{p}) \quad (5.18)$$

$$\langle j_{\mathbf{p}}^{(c)} j_{-\mathbf{p}}^{(c)} \rangle^{(g)} = \frac{1}{\pi Z^2 \bar{c} \bar{v}_{\rho,+}^2} \frac{-p_0^2(1 - \bar{\nu}_4 - 2\bar{\nu}_\rho) + \bar{c}^2 p^2(1 + \bar{\nu}_4 + 2\bar{\nu}_\rho)}{p_0^2 + \bar{c}^2 \bar{v}_\rho^2 p^2} + O(\mathbf{p})$$

From the WI (1.14) we see that

$$\hat{\Omega}_C(0, p_0) = 0, \quad \hat{D}(p, 0) = 0 \quad (5.19)$$

and this fixes the values of the constants  $A_c$  and  $A_j$  in (5.7), so that

$$\begin{aligned} \hat{\Omega}_C(\mathbf{p}) &= \frac{Z_3^2}{\pi Z^2 \bar{c} \bar{v}_{\rho,+}^2} [(1 + \bar{\nu}_4 - 2\bar{\nu}_\rho) + \bar{v}_\rho^2(1 - \bar{\nu}_4 + 2\bar{\nu}_\rho)] \frac{\bar{c}^2 p^2}{p_0^2 + \bar{v}_\rho^2 \bar{c}^2 p^2} + o(\mathbf{p}) \\ \hat{D}(\mathbf{p}) &= \frac{\tilde{Z}_3^2}{\pi Z^2 \bar{c} \bar{v}_{\rho,+}^2 v_\rho^2} [(1 + \bar{\nu}_4 + 2\bar{\nu}_\rho) + \bar{v}_\rho^2(1 - \bar{\nu}_4 - 2\bar{\nu}_\rho)] \frac{p_0^2}{p_0^2 + \bar{v}_\rho^2 \bar{c}^2 p^2} + o(\mathbf{p}) \end{aligned} \quad (5.20)$$

If we insert (5.13) in the previous equations, we get, for the susceptibility (1.9) and the Drude weight (1.12), the values

$$\begin{aligned} \kappa &= \frac{(1 - \bar{\nu}_4 - 2\bar{\nu}_\rho)^2}{\pi \bar{c} \bar{v}_{\rho,+}^2 \bar{v}_\rho^2} [(1 + \bar{\nu}_4 - 2\bar{\nu}_\rho) + \bar{v}_\rho^2(1 - \bar{\nu}_4 + 2\bar{\nu}_\rho)] = \frac{\bar{K}}{\pi \bar{c} \bar{v}_\rho} \\ D &= \frac{\bar{c}(1 + \bar{\nu}_4 - 2\bar{\nu}_\rho)^2}{\pi \bar{v}_{\rho,+}^2 \bar{v}_\rho^2} [(1 + \bar{\nu}_4 + 2\bar{\nu}_\rho) + \bar{v}_\rho^2(1 - \bar{\nu}_4 - 2\bar{\nu}_\rho)] = \frac{\bar{K} \bar{c} \bar{v}_\rho}{\pi} \end{aligned} \quad (5.21)$$

where

$$\bar{K} = \frac{(1 - 2\bar{\nu}_\rho)^2 - \bar{\nu}_4^2}{\bar{v}_{\rho+} \bar{v}_{\rho-}} = \sqrt{\frac{(1 - \bar{\nu}_4) - 2\bar{\nu}_\rho}{(1 - \bar{\nu}_4) + 2\bar{\nu}_\rho}} \sqrt{\frac{(1 + \bar{\nu}_4) - 2\nu_\rho}{(1 + \bar{\nu}_4) + 2\bar{\nu}_\rho}} \quad (5.22)$$



so that

$$\frac{\kappa}{D} = \frac{1}{\bar{c}^2 \bar{v}_\rho^2} \quad (5.23)$$

and this completes the proof of Theorem 1.2.

**Remark** We are unable to see if  $\bar{c}\bar{v}_\rho = v_F(1 + \bar{\delta})\bar{v}_\rho$  coincides with the velocity  $cv_\rho = v_F(1 + \tilde{\delta})v_\rho$  appearing in the two-point function asymptotic behavior (3.34), with  $v_\rho$  given (see (3.14)) by

$$v_\rho = \frac{((1 + \nu_4)^2 - 4\nu_\rho^2)}{((1 - \nu_4)^2 - 4\nu_\rho^2)} \quad (5.24)$$

with  $\nu_\rho$  and  $\nu_4$  defined as in (3.53). In fact, it is easy to see that  $\tilde{\delta}$  is equal to  $\bar{\delta}$  at the first order and this is true also for  $\bar{v}_\rho$  and  $v_\rho$  by (5.6), (5.12) and (3.53); however, our arguments are not able to exclude that the two velocities are different. Moreover, (5.22) and (3.44) imply that  $\bar{K} = K$  at first order, but they also could be different. Note that the equality of  $\bar{K}$  and  $K$ , would imply that  $\kappa = K/v$ , with  $v = \bar{c}\bar{v}_\rho$  being the charge velocity, a relation proposed in [17] which, together with (1.22) and (1.24), would allow the exact determination of the exponents in terms of the susceptibility and the Drude weight.

## A The $g_1$ map

Let us consider the following map on the complex plane:

$$g_{n+1} = g_n - a_n g_n^2 \quad (A.1)$$

where  $a_n$  is a sequence depending on  $g_0$ , such that, if  $|g_0|$  is small enough,

$$a_n = a + \sigma_n, \quad |\sigma_n| \leq c_0 |g_0|, \quad (A.2)$$

for some positive constants  $a$  and  $c_0$ . We want to study the trajectory of the map (A.1), under the condition that

$$g_0 \in D_{\varepsilon, \delta} = \{z \in \mathbb{C} : |z| < \varepsilon, |\text{Arg}(z)| \leq \pi - \delta\}, \quad \delta \in (0, \pi/2) \quad (A.3)$$

We shall first study the properties of a sequence  $\tilde{g}_n$ , which turns out to be a good approximation of  $g_n$ . Let us define:

$$A_n = \frac{1}{n} \sum_{k=0}^{n-1} a_k \quad (A.4)$$

**Lemma A.1** *Given  $\delta \in (0, \pi/2)$ , there exists  $\varepsilon_0(\delta)$  such that, if  $\varepsilon \leq \varepsilon_0(\delta)$  and  $g_0 \in D_{\varepsilon, \delta}$ , the sequence*

$$\tilde{g}_n = \frac{g_0}{1 + g_0 n A_n} \quad (A.5)$$

*at any step  $n \geq 0$  is well defined and does not exit the larger domain  $D_{\varepsilon_1, \delta_1}$ , for  $\varepsilon_1 = 2\varepsilon/(\sin \delta)$  and  $\delta_1 = \delta/2$ .*

**Proof** - First of all, we choose  $\varepsilon$  so that

$$c_0 \varepsilon \leq a/2 \quad \Rightarrow \quad a/2 \leq \Re a_n \leq 3a/2, \quad |\Im a_n| \leq c_0 |g_0| \quad (A.6)$$

where  $c_0$  is the constant defined in (A.2); we can write

$$A_n = \alpha_n + i\beta_n, \quad \alpha_n \geq a/2, \quad |\beta_n| \leq c_0 |g_0|. \quad (A.7)$$

Define  $\tilde{z}_n := 1 + g_0 n A_n := 1 + g_0 n \alpha_n + \tilde{w}_n$ ; then, if  $g_0 \in D_{\varepsilon, \delta}$ ,

$$|1 + g_0 n \alpha_n| \geq \max \left\{ \sin \delta, \frac{\sin \delta}{3} (1 + |g_0| n \alpha_n) \right\} \quad (\text{A.8})$$

In fact, it is trivial to show that  $|1 + g_0 n \alpha_n| \geq \sin \delta$ ; on the other hand, if  $|g_0| n \alpha_n \geq 2$ ,

$$|1 + g_0 n \alpha_n| \geq |g_0| n \alpha_n - 1 = (|g_0| n \alpha_n + 2|g_0| n \alpha_n - 3)/3 \geq (|g_0| n \alpha_n + 1)/3$$

By using (A.8), we get

$$\frac{|\tilde{w}_n|}{|1 + g_0 n \alpha_n|} \leq \frac{6c_0}{a \sin \delta} |g_0|. \quad (\text{A.9})$$

It follows that, if  $\varepsilon$  is small enough,

$$|\tilde{z}_n| \geq \frac{1}{2} \sin \delta \quad (\text{A.10})$$

so that, in particular, the definition (A.5) is meaningful.

Now we want to prove that  $\tilde{g}_n \in D_{\varepsilon_1, \delta_1}$ , with  $\varepsilon_1 = 2\varepsilon/(\sin \delta)$  and  $\delta_1 = \delta/2$ , if  $\varepsilon$  is small enough. Let  $g_0 = \rho_0 e^{i\theta_0}$ ; by using (A.8) and (A.9), we see that, if  $\varepsilon$  is small enough,

$$|\tilde{g}_n| \leq \frac{2|g_0|}{|1 + \alpha_n g_0 n|} \leq \frac{2\varepsilon}{\sin \delta}; \quad (\text{A.11})$$

besides it is easy to see that

$$\left| \text{Arg} \left( \frac{g_0}{1 + \alpha_n g_0 n} \right) \right| = \left| \text{Arg} \left( \frac{\rho_0}{e^{-i\theta_0} + \alpha_n \rho_0 n} \right) \right| \leq |\theta_0| \leq \pi - \delta.$$

Then, since  $\tilde{g}_n = \frac{\rho_0}{e^{-i\theta_0} + \alpha_n \rho_0 n} (1 + w_n)$ , with  $w_n$  of order  $g_0$ , for  $\varepsilon$  small enough,

$$|\text{Arg}(\tilde{g}_n)| \leq \pi - \delta/2 \quad (\text{A.12})$$

■

**Proposition A.2** *Given  $\delta \in (0, \pi/2)$ , there exists  $\varepsilon_0(\delta)$ , such that, if  $\varepsilon \leq \varepsilon_0(\delta)$  and  $g_0 \in D_{\varepsilon, \delta}$ , then*

$$g_n \in D_{\varepsilon_2, \delta_2}, \quad \varepsilon_2 = \frac{3\varepsilon}{\sin \delta}, \quad \delta_2 = \frac{\delta}{4} \quad (\text{A.13})$$

Moreover, if  $\tilde{g}_n$  is defined as in (A.5),

$$|g_n - \tilde{g}_n| \leq |\tilde{g}_n|^{3/2} \quad (\text{A.14})$$

**Proof** - We shall proceed by induction on the condition (A.14), which is true for  $n = 0$ . Suppose that it is true for  $n \leq N$ ; then, by using (A.11) and (A.12), we see that, if  $\varepsilon$  is small enough and  $n \leq N$ ,

$$|g_n| \leq 3|\tilde{g}_n|/2 \leq 3\varepsilon/\sin \delta, \quad |\text{Arg}(g_n)| \leq \pi - \delta/4 \quad (\text{A.15})$$

which proves (A.13). Moreover, by (A.1), if  $\varepsilon$  is small enough,

$$|g_{N+1}| \leq 2|g_N| \leq 3|\tilde{g}_N| \quad (\text{A.16})$$

Note now that

$$\frac{1}{g_{n+1}} - \frac{1}{g_n} = \frac{a_n}{1 - a_n g_n} = a_n + a_n^2 g_n + \Delta_n = \frac{1}{g_{n+1}} - \frac{1}{g_n} + a_n^2 g_n + \Delta_n \quad (\text{A.17})$$

where  $\Delta_n$  is a quantity which can be bounded by  $c_1|g_n|^2$ , for some constant  $c_1$ . We can rewrite (A.17) in the form

$$\frac{1}{g_{n+1}} - \frac{1}{\tilde{g}_{n+1}} = \frac{1}{g_n} - \frac{1}{\tilde{g}_n} + a_n^2 g_n + \Delta_n \quad (\text{A.18})$$

By using (A.6), (A.8), (A.9), (A.15), (A.16) and (A.18), we get, if  $\varepsilon$  is small enough,

$$\begin{aligned} |g_{N+1} - \tilde{g}_{N+1}| &= |g_{N+1}| |\tilde{g}_{N+1}| \left| \frac{1}{g_{N+1}} - \frac{1}{\tilde{g}_{N+1}} \right| \\ &\leq 3|\tilde{g}_N| |\tilde{g}_{N+1}| \sum_{n=0}^N [6a^2|\tilde{g}_n| + \frac{9}{4}c_1|\tilde{g}_n|^2] \leq c_2|\tilde{g}_N|^{3/2} \frac{|g_0|^{1/2}}{(1 + \frac{a}{2}|g_0|N)^{1/2}} \sum_{n=0}^N \frac{|g_0|}{1 + \frac{a}{2}|g_0|n} \\ &\leq |\tilde{g}_N|^{3/2} \frac{c_3|g_0|^{1/2}}{(1 + \frac{a}{2}|g_0|N)^{1/2}} \log \left( 1 + \frac{a}{2}|g_0|N \right) \leq |\tilde{g}_N|^{3/2} \end{aligned} \quad (\text{A.19})$$

where  $c_2$  and  $c_3$  are two suitable constants. ■

## B Symmetries of the Effective Model and RG Flow

The RG analysis of the effective model will be done by exploiting some symmetry properties of a more general model, obtained by adding to the interaction (3.3) the term  $g_3 V_3(\psi)$ , with

$$V_3(\psi) = \frac{1}{2} \sum_{\omega, s} \int d\mathbf{x} d\mathbf{y} h(\mathbf{x} - \mathbf{y}) \psi_{\mathbf{x}, \omega, s}^+ \psi_{\mathbf{x}, \omega, -s}^+ \psi_{\mathbf{y}, -\omega, s}^- \psi_{\mathbf{y}, -\omega, -s}^- \quad (\text{B.1})$$

The integration of the positive (ultraviolet) scales  $N, N-1, \dots, 0$  is essentially identical to the one described in §2 of [29] or §3 of [39], for the spinless case; it does not need any localization procedure.

The integration of the infrared scales is done in a way similar to the one in the Hubbard model described in §2. However, before starting the multiscale IR integration, we have to perform some technical operations, which will make possible to compare the flow of the running couplings with that of the Hubbard model.

After the integration of the UV scales up to  $j = 1$ , the free measure propagator is given by  $g_{\text{D}, \omega}^{[l, 1]}(\mathbf{x})$ , defined as in (3.2) with  $N = 1$ . In this expression, the velocity  $c$  has the role of the Fermi velocity  $v_F$  of the Hubbard model. In order to match the asymptotic behavior of the two models, we can not choose  $c = v_F$ ; for this reason we introduced the parameter  $\delta$ . However, it is not possible to compare the RG flows of the two models if the two velocities are different; hence, we have to move from the free measure to the interaction the term proportional to  $\delta$ . Moreover, since also the cutoff function  $\chi^{[l, 1]}(|(k_0, ck)|)$  depends on  $\delta$ , we have to “modify” it in  $\chi_{[l, 1]}(|(k_0, v_F k)|)$ .

The simplest way of performing these operations without introducing spurious singularities is the following one. We start with a free measure of the form

$$P(d\psi^{[l, 1]}) = \mathcal{N}^{-1} \exp \left\{ -\frac{Z}{L^2} \sum_{\mathbf{k}, \omega, s} C_l(\mathbf{k}) [-ik_0 + \omega v_F (1 + \delta)k] \hat{\psi}_{\mathbf{k}, \omega, s}^{+[l, 1]} \psi_{\mathbf{k}, \omega, s}^{-[l, 1]} \right\} \quad (\text{B.2})$$

where  $C_l(\mathbf{k}) = \chi^{[l, 1]}(|(k_0, ck)|)^{-1}$ . We can move to the scale 1 effective interaction the term

$$-\delta \frac{Z}{L^2} \sum_{\mathbf{k}, \omega, s} \omega v_F k \hat{\psi}_{\mathbf{k}, \omega, s}^{+[l, 1]} \psi_{\mathbf{k}, \omega, s}^{-[l, 1]} = -\delta V_\delta(\psi), \quad \text{with}$$

$$V_\delta(\psi) = \sum_{\omega,s} \int d\mathbf{x} \psi_{\mathbf{x},\omega,s}^+ (i\omega v_F \partial_x) \psi_{\mathbf{x},\omega,s}^- \quad (\text{B.3})$$

The new free measure differs from (B.2) because  $\delta$  is multiplied by  $u_l(\mathbf{k}) = 1 - \chi^{[l,1]}(|(k_0, ck)|)$ . On the other hand, since  $\delta$  will be chosen of order  $\lambda$ ,  $\chi^{[l,1]}(|(k_0, ck)|)$  and  $\chi^{[l,1]}(|(k_0, v_F k)|)$  differ only for values of  $\mathbf{k}$  of size  $\gamma$  or  $\gamma^l$  and

$$u_l(\mathbf{k}) = 0, \quad \text{if } \chi^{[l+1,0]}(|(k_0, v_F k)|) > 0 \quad (\text{B.4})$$

Hence, we can write

$$\chi^{[l,1]}(|(k_0, ck)|) = \bar{\chi}^{(1)}(\mathbf{k}) + \chi^{[l+1,0]}(|(k_0, v_F k)|) + \bar{\chi}^{(l)}(\mathbf{k}) \quad (\text{B.5})$$

with  $\bar{\chi}^{(1)}(\mathbf{k})$  and  $\bar{\chi}^{(l)}(\mathbf{k})$  smooth functions, whose support is on values of  $\mathbf{k}$  of size  $\gamma$  or  $\gamma^l$ , respectively; moreover, if we define

$$\tilde{C}_l(\mathbf{k}) = \left[ \chi^{[l+1,0]}(|(k_0, v_F k)|) + \bar{\chi}^{(l)}(\mathbf{k}) \right]^{-1} \quad (\text{B.6})$$

then  $\tilde{C}_l(\mathbf{k}) = 1$ , if  $1 \geq |(k_0, v_F k)| \geq \gamma^{l+1}$  and  $\tilde{C}_l(\mathbf{k})^{-1} \bar{\chi}^{(l)}(\mathbf{k}) \leq 1$ . It follows that the free measure  $P(d\psi^{[l,1]})$  can be written as  $P(d\bar{\psi}^{(1)})P(d\tilde{\psi}^{[l,0]})$ , where  $\bar{\psi}^{(1)}$  is a field whose covariances has the same scale properties of  $\psi^{(1)}$ , while

$$P(d\tilde{\psi}^{[l,0]}) = \mathcal{N}^{-1} \exp \left\{ -\frac{Z}{L^2} \sum_{\mathbf{k},\omega,s} \tilde{C}_l(\mathbf{k}) [-ik_0 + \omega v_F (1 + u_l(\mathbf{k})\delta)k] \tilde{\psi}_{\mathbf{k},\omega,s}^{+[l,0]} \tilde{\psi}_{\mathbf{k},\omega,s}^{-[l,0]} \right\} \quad (\text{B.7})$$

The integration of the single scale field  $\bar{\psi}^{(1)}$  can be done without any problem. At this point, we start the multiscale integration of the field  $\tilde{\psi}^{[l,0]}$ , by performing the effective potential localization and the free measure renormalization as in the Hubbard model. Thanks to the support properties of  $u_l(\mathbf{k})$ , the steps from  $j = 0$  to  $j = l+1$  will give the same result we should get if the propagator of  $\tilde{\psi}^{[l,0]}$  were equal to

$$\frac{1}{Z} \frac{1}{L^2} \sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \frac{\chi_{l,0}(|(k_0, v_F k)|)}{-ik_0 + \omega v_F k}$$

This means that the renormalized single scale propagator will have the form (2.38), corresponding to the leading behavior of the single scale propagator in the Hubbard model. This property is not true only in the last step,  $j = l$ , but this is not a problem, since we have to study the RG flow at fixed  $j$  and  $l \rightarrow -\infty$  and, moreover, the contribution of the IR scale fluctuations to the Schwinger functions at fixed space-time coordinates vanishes as  $l \rightarrow -\infty$ .

Let us now analyze in more detail the RG flow of the effective model for  $h \leq 0$ . The main difference with respect to the Hubbard model is that (2.24) has to be replaced by

$$\begin{aligned} \tilde{V}^{(j)}(\sqrt{Z_j}\psi) &= g_{1,\perp,j} F_{1,\perp}(\sqrt{Z_j}\psi) + g_{\parallel,j} F_{\parallel}(\sqrt{Z_j}\psi) + \\ &+ g_{\perp,j} F_{\perp}(\sqrt{Z_j}\psi) + g_{3,j} F_3(\sqrt{Z_j}\psi) + g_{4,j} F_4(\sqrt{Z_j}\psi) + \delta_j V_\delta(\sqrt{Z_j}\psi) \end{aligned} \quad (\text{B.8})$$

where the functions  $F_\alpha(\psi)$  are defined as the functions  $V_\alpha(\psi)$  of (3.4) with  $h(\mathbf{x} - \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$ ; the absence of local terms proportional to  $\psi^+ \psi^-$  is a consequence of the oddness in  $\mathbf{k}$  of the free propagator. The running couplings verify equations of the form

$$\begin{aligned} g_{\alpha,h-1} - g_{\alpha,h} &= B_\alpha^{(h)}(\vec{g}_h, \delta_h, \dots, \vec{g}_0, \delta_0, \vec{g}, \delta) \\ \delta_{h-1} - \delta_h &= B_\delta^{(h)}(\vec{g}_h, \delta_h, \dots, \vec{g}_0, \delta_0, \vec{g}, \delta) \end{aligned} \quad (\text{B.9})$$

where  $\alpha = 1, \perp, \parallel, 3, 4$  and  $\vec{g}_j = (g_{1,\perp,j}, g_{3,j}, g_{\parallel,j}, g_{\perp,j}, g_{4,j})$ . Note that the functions  $B_\alpha^{(h)}$  and  $B_\delta^{(h)}$  are of the second order in their arguments; in the case of  $B_\delta^{(h)}$ , this follows from the

structure of  $\tilde{V}(\psi)$  (see (3.4)), which does not allow to build Feynman graphs of the first order in  $\vec{g}$ . For the same reason

$$\delta_0 = \delta + O(\varepsilon_0^2), \quad \varepsilon_0 = \max\{|\vec{g}|, |\delta|\} \quad (\text{B.10})$$

and this relations can be inverted, if  $\varepsilon_0$  is small enough.

There are some symmetries which is important to exploit. For notational simplicity, we will write  $(G_{1,\perp}, G_3, G_{\parallel}, G_{\perp}, G_4, \Delta)$  or  $(\vec{G}, \Delta)$  in place of  $(\vec{g}_h, \delta_h, \dots, \vec{g}_0, \delta_0, \vec{g}, \delta)$ .

a. *Spin  $U(1)$ .* Free measure and interactions are invariant under the transformation

$$\psi_{\mathbf{x},\omega,s}^{\varepsilon} \rightarrow e^{i\varepsilon\alpha_s} \psi_{\mathbf{x},\omega,s}^{\varepsilon}$$

where  $\alpha_s$  is a spin-dependent angle. This means that the only possible local effective interactions must have as many  $\psi_s^+$  as  $\psi_s^-$ , for each given  $s$ . Therefore, all the allowed local quartic interactions are the ones listed in (3.4); it is also clear from the symmetries  $\omega \rightarrow -\omega$  and  $s \rightarrow -s$  that they must occur in the same linear combinations.

b. *Particle-hole symmetry.* The free measure is invariant if the particle hole switching involves the spin  $s'$  only, i.e.  $\hat{\psi}_{\mathbf{k},\omega,s}^{\varepsilon} \rightarrow \hat{\psi}_{-ss'\mathbf{k},\omega,s}^{-ss'\varepsilon}$ ; interactions are not invariant and we find:

$$B_{1,\perp}^{(h)}(G_{1,\perp}, G_3, G_{\parallel}, G_{\perp}, G_4, \Delta) = -B_3^{(h)}(-G_3, -G_{1,\perp}, G_{\parallel}, -G_{\perp}, -G_4, \Delta) \quad (\text{B.11})$$

$$\left. \begin{aligned} B_{\parallel}^{(h)}(G_{1,\perp}, G_3, G_{\parallel}, G_{\perp}, G_4, \Delta) &= B_{\parallel}^{(h)}(-G_3, -G_{1,\perp}, G_{\parallel}, -G_{\perp}, -G_4, \Delta) \\ B_{\perp}^{(h)}(G_{1,\perp}, G_3, G_{\parallel}, G_{\perp}, G_4, \Delta) &= -B_{\perp}^{(h)}(-G_3, -G_{1,\perp}, G_{\parallel}, -G_{\perp}, -G_4, \Delta) \\ B_4^{(h)}(G_{1,\perp}, G_3, G_{\parallel}, G_{\perp}, G_4, \Delta) &= -B_4^{(h)}(-G_3, -G_{1,\perp}, G_{\parallel}, -G_{\perp}, -G_4, \Delta) \end{aligned} \right\} \quad (\text{B.12})$$

$$B_{\Delta}^{(h)}(G_{1,\perp}, G_3, G_{\parallel}, G_{\perp}, G_4, \Delta) = B_{\Delta}^{(h)}(-G_3, -G_{1,\perp}, G_{\parallel}, -G_{\perp}, -G_4, \Delta) \quad (\text{B.13})$$

c. *Chiral  $U(1)$ .* The free measure is invariant under the transformation

$$\hat{\psi}_{\mathbf{k},\omega,s}^{\varepsilon} \rightarrow e^{i\varepsilon\alpha_{\omega}} \hat{\psi}_{\mathbf{k},\omega,s}^{\varepsilon} \quad (\text{B.14})$$

for  $\alpha_{\omega}$  a chirality-dependent angle. All the interactions are  $U(1)$  invariant, but for  $V_3$ ; if  $g_3 = 0$ , then an interaction  $V_3$  won't be generated by the flow. This fact can be seen also graph by graph. Indeed, in the graphs for  $B_{\parallel}$ ,  $B_{\perp}$ ,  $B_4$  and  $B_{\delta}$ , the number of the half-lines  $\psi_{\omega}^+$  has to equal the number of the half-lines  $\psi_{\omega}^-$  (regardless the spin label); this can only happen when there is an even number of interactions  $V_3$ . Therefore these three beta functions are even in  $G_3$  and then, by (B.12) and (B.13), also even in  $G_{1,\perp}$ ; hence, if  $\alpha = \parallel, \perp, 4, \delta$ ,

$$B_{\alpha}^{(h)}(G_{1,\perp}, G_3, G_{\parallel}, G_{\perp}, G_4, \Delta) = \bar{B}_{\alpha}^{(h)}(G_{1,\perp}^2, G_3^2, G_{\parallel}, G_{\perp}, G_4, \Delta) \quad (\text{B.15})$$

where  $G_{\alpha}^2$  denotes the tensor  $\{g_{\alpha,j}g_{\alpha,j'}\}_{j,j' \geq h}$ . By a similar argument,  $B_3^{(h)}$  has to be odd in  $G_3$  and  $B_1^{(h)}$  even; then, by (B.11),  $B_1^{(h)}$  is also odd in  $G_1$  and  $B_3^{(h)}$  even, so that

$$\begin{aligned} B_{1,\perp}^{(h)}(G_{1,\perp}, G_3, G_{\parallel}, G_{\perp}, G_4, \Delta) &= G_{1,\perp} \bar{B}^{(h)}(G_{1,\perp}^2, G_3^2, G_{\parallel}, G_{\perp}, G_4, \Delta) \\ B_3^{(h)}(G_{1,\perp}, G_3, G_{\parallel}, G_{\perp}, G_4, \Delta) &= G_3 \bar{B}^{(h)}(G_3^2, G_{1,\perp}^2, G_{\parallel}, -G_{\perp}, -G_4, \Delta) \end{aligned} \quad (\text{B.16})$$

where  $G_{\alpha} \bar{B}_{\alpha}^{(h)}$  is a shorthand for  $\sum_{j \geq h} g_{\alpha,j} B_{\alpha}^{(h,j)}$ .

In this way we have found two invariant surfaces in the space of the interaction parameters  $(\vec{g}, \delta)$ :

$$\mathcal{C}_1 = \{\vec{g}, \delta : g_{1,\perp} = 0\} \quad \mathcal{C}_3 = \{\vec{g}, \delta : g_3 = 0\}$$

d. *Spin*  $SU(2)$ . It is convenient to rewrite the interaction as

$$\begin{aligned} \tilde{V}(\psi) = & g_{1,\perp} (V_{1,\perp}(\psi) - V_{\parallel}(\psi)) + (g_{\parallel} + g_{1,\perp} - g_{\perp}) V_{\parallel}(\psi) + \\ & + g_{\perp} (V_{\perp}(\psi) + V_{\parallel}(\psi)) + g_3 V_3(\psi) + g_4 V_4(\psi) + \delta_h V_{\delta}(\psi) \end{aligned} \quad (\text{B.17})$$

It is evident that  $V_{1,\perp} - V_{\parallel}$ ,  $V_{\perp} + V_{\parallel}$ ,  $V_3$ ,  $V_4$  and  $V_{\delta}$ , as well as the free measure, are invariant under the transformation of the fields

$$\hat{\psi}_{\mathbf{k},\omega,s}^- \rightarrow \sum_{s'} U_{s,s'} \hat{\psi}_{\mathbf{k},\omega,s'}^-, \quad \hat{\psi}_{\mathbf{k},\omega,s}^+ \rightarrow \sum_{s'} \hat{\psi}_{\mathbf{k},\omega,s'}^+ U_{s',s}^\dagger$$

for  $U \in SU(2)$ . While  $V_{\parallel}$  isn't: if  $g_{\parallel} + g_1 - g_{\perp} = 0$  it will remain zero. Thus we find four other invariant surfaces:

$$C_{1,+} = \{\vec{g}, \delta : g_{1,\perp} = g_{\perp} - g_{\parallel}\}, \quad C_{3,+} = \{\vec{g}, \delta : g_3 = g_{\perp} + g_{\parallel}\}$$

$$C_{1,-} = \{\vec{g}, \delta : -g_{1,\perp} = g_{\perp} - g_{\parallel}\}, \quad C_{3,-} = \{\vec{g}, \delta : -g_3 = g_{\perp} + g_{\parallel}\}$$

(in fact  $C_{1,-}$ ,  $C_{3,+}$  and  $C_{3,-}$  are obtained from  $C_{1,+}$  through (B.11), (B.12), (B.15) and (B.16)).  $C_{3,+}$  is also called Fowler invariant (see [12], page 220).

e. *Vector-Axial Symmetry*. All the terms in  $\tilde{V}(\psi)$ , but  $V_1(\psi)$  and  $V_3(\psi)$ , are invariant under the transformation

$$\psi_{\mathbf{x},\omega,s}^\varepsilon \rightarrow e^{i\varepsilon\vartheta_{\omega,s}} \psi_{\mathbf{x},\omega,s}^\varepsilon$$

therefore the surface  $g_{1,\perp} = g_3 = 0$  is invariant.

Finally we consider the flow of  $Z_h$  and the renormalization constant  $Z_h^{(1)}$  associated with the density operator  $\rho_{\mathbf{x},\omega,s} = \psi_{\mathbf{x},\omega,s}^+ \psi_{\mathbf{x},\omega,s}^-$  in the generating functional (3.1);  $Z_h^{(1)}$  is defined as  $Z_h^{(1,C)}$  in (2.27). It is easy to see, by using the symmetry properties of the model as before, that  $Z_{h-1}/Z_h = 1 + B_z^{(h)}(\vec{G}, \Delta)$  and  $Z_{h-1}^{(1)}/Z_h^{(1)} = 1 + B_\rho^{(h)}(\vec{G}, \Delta)$ , with  $B_\alpha^{(h)}(\vec{G}, \Delta) = \bar{B}_\alpha^{(h)}(G_1^2, G_3^2, G_{\parallel}, G_{\perp}, G_4, \Delta)$  for  $\alpha = z, \rho$ . Hence

$$\frac{Z_{h-1}^{(1)}}{Z_{h-1}} = \frac{Z_h^{(1)}}{Z_h} [1 + \tilde{B}^{(h)}(\vec{G}, \Delta)] \quad (\text{B.18})$$

with

$$\tilde{B}^{(h)}(\vec{G}, \Delta) = \bar{B}_2^{(h)}(G_{1,\perp}^2, G_3^2, G_{\parallel}, G_{\perp}, G_4, \Delta) \quad (\text{B.19})$$

## C Vanishing of the Beta Function

We recall the main ideas of the proof of (2.45) and (2.83) (see [22],[23],[24],[33] for more details). We consider the model (3.1) with  $g_{1,\perp} = 0$ ;  $\delta$ ,  $g_{\parallel}$  and  $g_{\perp}$  are small but arbitrary parameters. We take the limit  $N \rightarrow \infty$  at fixed  $l$ ; if  $|\mathbf{k}| = \gamma^l$  (so that, in particular,  $\hat{f}_l(\mathbf{k}) = 1$ )

$$\hat{S}_\omega(\mathbf{k}) \equiv \langle \hat{\psi}_{\mathbf{k},\omega,s}^- \hat{\psi}_{\mathbf{k},\omega,s}^+ \rangle_l = \frac{1}{Z_l D_{l,\omega}(\mathbf{k})} [1 + W_2^{(l)}(\mathbf{k})] \quad (\text{C.1})$$

where  $\langle \cdot \rangle_l$  denotes the expectation with propagator (3.2),

$$D_l(\mathbf{k}) = -ik_0 + \omega v_F(1 + \delta_l)k$$

and

$$|W_2^{(l)}(\mathbf{k})| \leq C(\varepsilon_l^2 + \bar{g}_l \gamma^{\vartheta l}) \quad (\text{C.2})$$

with  $\bar{g}_l = \max_{h \geq l} \max\{|g_{2,h}|, |g_{4,h}|\}$  and  $\varepsilon_l = \max_{j \geq l} \max\{\bar{g}_j, |\delta_j|\}$ , Note that we have included in  $D_l(\mathbf{k})$  a correction of the free Fermi velocity due to the local term  $\delta_l V_\delta$ , so that  $W_2^{(l)}(\mathbf{k})$  does not contain terms of order  $\varepsilon_l$ , except those associated to irrelevant terms. Moreover, by the analogue of (112) of [22]

$$\langle \hat{\rho}_{\mathbf{p},\omega,s} \hat{\psi}_{\mathbf{k},\omega,s}^- \hat{\psi}_{\mathbf{k}-\mathbf{p},\omega,s}^+ \rangle_l = Z_l^{(1)} \hat{S}_\omega(\mathbf{k}) \hat{S}_\omega(\mathbf{k}-\mathbf{p}) [1 + W_{2,1}^{(l)}(\mathbf{k})] \quad (\text{C.3})$$

where  $|W_{2,1}^{(l)}(\mathbf{k})| \leq C(\varepsilon_l^2 + \bar{g}_l \gamma^{\vartheta l})$ . In §3.2 we have written the WI directly in the  $l \rightarrow -\infty$  limit, but they are true also when  $l$  is finite, up to corrections studied in detail in §4 of [22] and §4 of [23]; using the bounds in such papers for the corrections we derive from the finite  $l$  analogue of (3.6)

$$D_N(\mathbf{p}) \langle \hat{\rho}_{\mathbf{p},\omega,s} \hat{\psi}_{\mathbf{k},\omega,s}^- \hat{\psi}_{\mathbf{k}-\mathbf{p},\omega,s}^+ \rangle_l = \hat{S}_\omega(\mathbf{k}-\mathbf{p}) - \hat{S}_\omega(\mathbf{k}) + D_N(\mathbf{p}) R_l(\mathbf{k}, \mathbf{p}) \quad (\text{C.4})$$

where  $\mathbf{k}$  and  $\mathbf{k}-\mathbf{p}$  are of size  $\gamma^l$ ,

$$D_N(\mathbf{k}) = -ik_0 + \omega_{VF}(1 + \delta)k$$

and

$$|R_l(\mathbf{k}, \mathbf{p})| \leq C\gamma^{-2l} \frac{Z_l^{(1)}}{Z_l^2} (\varepsilon_l^2 + \bar{g}_l \gamma^{\vartheta l})$$

Hence, if we put  $\mathbf{k} = \bar{\mathbf{k}}$  and  $\mathbf{p} = 2\bar{\mathbf{k}}$ , with  $|\bar{\mathbf{k}}| = \gamma^l$ , we get  $D_N(\bar{\mathbf{k}})[Z_l^{(1)}/Z_l] = D_l(\bar{\mathbf{k}})[1 + \Delta(\bar{\mathbf{k}})]$ , with  $|\Delta(\mathbf{k})| \leq C(\varepsilon_l^2 + \bar{g}_l \gamma^{\vartheta l})$ , which implies that  $\delta_l = \delta + O(\varepsilon_l^2)$  and  $Z_l^{(1)}/Z_l = 1 + O(\varepsilon_l^2)$ . On the other hand, the value of  $\delta_h$  is independent of the infrared cutoff, if  $h \geq l+1$ , and  $|\delta_{l+1} - \delta_l| \leq C\varepsilon_l^2$ . It follows that, for any  $j \in [l, N]$  and uniformly in the cutoffs,

$$\delta_j = \delta + O(\varepsilon_j^2) \quad (\text{C.5})$$

$$\frac{Z_j^{(1)}}{Z_j} = 1 + O(\varepsilon_j^2) \quad (\text{C.6})$$

In the same way, we can also use the the Schwinger-Dyson equation combined with the Ward Identities with a finite infrared cut-off; we get (3.24) with a correction term, which can be bounded as in §4 of [23]. If we restrict the resulting expression to the four point function with momenta at the infrared scale, we get, see [23],[24]

$$g_{\parallel,j} = g_{\parallel} + O(\varepsilon_j^2), \quad g_{\perp,j} = g_{\perp} + O(\varepsilon_j^2), \quad g_{4,j} = g_4 + O(\varepsilon_j^2) \quad (\text{C.7})$$

Let us call  $b_\alpha^{(j)}(g_{\parallel}, g_{\perp}, g_4, \delta)$  the function which is obtained by  $B_\alpha^{(j)}(\vec{g}_h, \delta_h, \dots, \vec{g}_0, \delta_0, \vec{g}, \delta)$  (defined in (B.9)), by subtracting the contribution of the trees containing endpoints of scale greater than 0 and by putting  $(\vec{g}_j, \delta_j) = (0, 0, g_{\parallel}, g_{\perp}, g_4, \delta)$ ,  $\forall j = h, \dots, 0$  (the value of the r.c.c. is independent of the scale). Then, by Lemma 3.4 of [37], we get the bound:

$$\left| b_\alpha^{(j)}(g_{\parallel}, g_{\perp}, g_4, \delta) \right| \leq C[\max\{|g_{\parallel}|, |g_{\perp}|, |g_4|, |\delta|\}]^2 \gamma^{\vartheta j}, \quad \alpha = \parallel, \perp, 4, \delta \quad (\text{C.8})$$

In we define in a similar way the function  $\tilde{b}^{(j)}(g_{\parallel}, g_{\perp}, g_4, \delta)$  in terms of  $\tilde{B}^{(j)}(\vec{g}_j, \delta_j, \dots, \vec{g}_0, \delta_0, \vec{g}, \delta)$  (defined in (B.19)), (C.6) implies the bound

$$\left| \tilde{b}^{(j)}(g_{\parallel}, g_{\perp}, g_4, \delta) \right| \leq C[\max\{|g_{\parallel}|, |g_{\perp}|, |g_4|, |\delta|\}]^2 \gamma^{\vartheta j} \quad (\text{C.9})$$

The bounds (C.8) and (C.9) allow to show that, in the model (3.1), the infrared cut-off can be removed by an analysis very similar to the one in §2.3-2.4.

Let us now consider the Hubbard model. In (2.41) we have written its Beta function as sum of two terms, the second of which is asymptotically negligible, by (2.42); the first term, denoted in (2.41) by  $\beta_\alpha^{(j)}(\vec{g}_j, \delta_j; \dots; \vec{g}_0, \delta_0) \equiv \beta_\alpha^{(j)}(G_1, G_2, G_4, \Delta)$  (using a notation similar to that used after (B.9)) coincides with the Beta function of the effective model on the invariant surface  $\mathcal{C}_{1,+} \cap \mathcal{C}_3$ , if we subtract from it the contribution of the trees containing endpoints of scale greater than 0. Hence, by using (B.15) and (B.16), we get

$$\begin{aligned}\beta_1^{(j)}(G_1, G_2, G_4, \Delta) &= G_1 \bar{\beta}_{1\perp}^{(j)}(G_1^2, 0, G_2 - G_1, G_2, G_4, \Delta) \\ \beta_2^{(j)}(G_1, G_2, G_4, \Delta) &= \bar{\beta}_\perp^{(j)}(G_1^2, 0, G_2 - G_1, G_2, G_4, \Delta) \\ \beta_4^{(j)}(G_1, G_2, G_4, \Delta) &= \bar{\beta}_4^{(j)}(G_1^2, 0, G_2 - G_1, G_2, G_4, \Delta) \\ \beta_\delta^{(j)}(G_1, G_2, G_4, \Delta) &= \bar{\beta}_\delta^{(j)}(G_1^2, 0, G_2 - G_1, G_2, G_4, \Delta)\end{aligned}\tag{C.10}$$

where  $\bar{\beta}_\alpha^{(j)}(G_1^2, 0, G_2 - G_1, G_2, G_4, \Delta)$  denotes the value of  $\bar{B}_\alpha^{(j)}(G_1^2, 0, G_2 - G_1, G_2, G_4, \Delta)$ , after the subtraction of the trees containing endpoints of scale greater than 0. Therefore, if  $\alpha \neq 1$ , the contributions of order 0 and 1 in  $G_1$  of  $\beta_\alpha^{(j)}(\vec{G}, \Delta)$  are the same as the contributions of the same order of  $\bar{\beta}_\alpha^{(j)}(0, 0, G_2 - G_1, G_2, G_4, \Delta)$ . By using (C.8), one sees immediately that these contributions are of order  $\gamma^{\vartheta_j}$  as functions of  $j$ , so that (2.45) is proved. In the same way, using (C.9) and (B.19) we can prove (2.83).

This completes the proof of the boundedness of the RG flow in the Hubbard model, as well as in the effective model with  $g_{1,\perp} > 0$  and  $g_\parallel = g_\perp - g_{1,\perp}$ . Note that, in order to prove the boundedness of the flow of the spin-symmetric Hubbard model, we need information from a non spin symmetric model; in fact, we have derived (2.45) from the model (3.1) with  $g_\perp \neq g_\parallel$  and  $g_{1,\perp} = 0$ .

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